SOLVING THE PARABOLIC SINE-GORDON EQUATION WITH LINEARLY IMPLEMENTED ENERGY-STABLE CONVEX SPLITTING

Minhwan Ji and Jaemin Shin

Department of Mathematics, Chungbuk National University, Cheongju 28644, KOREA

Corresponding Author : Jaemin Shin, jmshin20@chungbuk.ac.kr

ABSTRACT

We introduce an energy-stable linear convex splitting method for the parabolic sine-Gordon equation. This linear scheme guarantees unique solvability and high computational efficiency. To improve temporal accuracy, we apply the convex splitting Runge–Kutta method, resulting in second-order accuracy. We prove that the first-order scheme satisfies the notable property of the discrete maximum principle, while our numerical experiments show that the second-order scheme exhibits only minor violations. Numerical experiments validate the method's accuracy, energy stability, and dynamic behavior. For spatial discretization, a conventional finite difference scheme with second-order accuracy is adopted.

INTRODUCTION

In [1], the parabolic sine-Gordon (PSG) equation was introduced as a parabolic version of the sine-Gordon equation as follows:

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} = \Delta\phi(\mathbf{x},t) - \frac{2\pi}{\epsilon^2}\sin(2\pi\phi(\mathbf{x},t)), \tag{1}$$

where $\phi(\mathbf{x},t)$ denotes the state variable on the bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$, and ϵ is a positive constant. This equation is the gradient flow of the energy functional

$$\mathcal{E}(\phi) = \int_\Omega \left[\frac{1}{2}|\nabla\phi|^2 + \frac{1}{\epsilon^2}\Big(1-\cos(2\pi\phi)\Big)\right] dx. \tag{2}$$

The evolution of ϕ according to this gradient flow ensures energy dissipation and maintains a maximum principle. Although it exhibits mean curvature flow similarly to the Allen–Cahn equation [2], the PSG equation is a multiphase system in which the energy attains its minimum at integer values.

NUMERICAL METHOD

We develop an unconditionally energy-stable scheme for the PSG equation by applying a convex splitting method [3]. In particular, the energy functional (2) is split into a contractive part and an expansive part (i.e., $\mathcal{E}(\phi) = \mathcal{E}_c(\phi) - \mathcal{E}_e(\phi)$) by defining

$$\mathcal{E}_c(\phi) = \int_\Omega \left(\frac{1}{2} |\nabla\phi|^2 + \frac{2\pi^2}{\epsilon^2}\phi^2 \right) dx, \quad \mathcal{E}_e(\phi) = \int_\Omega \frac{1}{\epsilon^2}\Big(2\pi^2\phi^2 - (1 - \cos(2\pi\phi)) \Big) dx. \tag{3}$$

By treating $\mathcal{E}_c(\phi)$ implicitly and $\mathcal{E}_e(\phi)$ explicitly, we derive the semi-discretized linear scheme as follows:

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \Delta\phi^{n+1} - \frac{4\pi^2}{\epsilon^2}\phi^{n+1} + \frac{2\pi}{\epsilon^2}\Big(2\pi\phi^n - \sin(2\pi\phi^n) \Big), \tag{4}$$

where ϕ^n denotes the approximation of $\phi(\mathbf{x}, t_n)$. This scheme is proven to be uniquely solvable, energy stable, and to preserve the discrete maximum principle. Moreover, applying the convex splitting Runge–Kutta method [4] to the scheme (4), for $k = 1, 2, 3$,

$$\phi_k = \phi_0 - \Delta t \left(\sum_{l=1}^{k} r_{k,l} \left(\frac{4\pi^2}{\epsilon^2}\phi_l - \Delta\phi_l \right) - \sum_{l=0}^{k-1} r_{k,l+1} \left(\frac{4\pi^2}{\epsilon^2}\phi_l - \frac{2\pi}{\epsilon^2}\sin(2\pi\phi_l) \right) \right), \tag{5}$$

where $r_{k,l}$ are components of the Butcher tableau and $\phi^n = \phi_0$, $\phi^{n+1} = \phi_3$. This approach is extended to achieve second-order temporal accuracy with unconditional energy stability.

CONCLUDING REMARK

In this work, we have proposed a linear convex splitting scheme for the parabolic sine–Gordon equation, which is based on the global boundedness of its bulk free energy. This scheme ensures high computational efficiency, unique solvability, and energy stability. Moreover, we enhanced the temporal accuracy by applying a convex splitting Runge–Kutta method. Numerical experiments with varied initial conditions demonstrate that our proposed method attains second-order accuracy, unconditional energy stability, and exhibits the characteristic behavior of mean curvature flow. Notably, the second-order convex splitting Runge–Kutta method exhibits slight violations of the discrete maximum principle for larger time steps.

REFERENCES

1. Cheng, Xinyu, et al., “On a Parabolic Sine-Gordon Model”, *Numerical Mathematics: Theory, Methods and Applications*, Vol. 14, 2021, pp. 1068–1084.
2. Allen, S. M. and Cahn, J. W., “A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening”, *Acta metallurgica*, Vol. 27, 1979, pp. 1085–1095.
3. Eyre, David J., “An unconditionally stable one-step scheme for gradient systems”, *Unpublished article*, Vol. 6, 1998.
4. Shin, J.M., Lee, H.G. and Lee, J.Y., “Unconditionally stable methods for gradient flow using Convex Splitting Runge–Kutta scheme”, *Journal of Computational Physics*, Vol. 347, 2017, pp. 367–381.