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NEW APPROXIMATE FIXED POINT RESULTS FOR VARIOUS CYCLIC CONTRACTION OPERATORS ON E-METRIC SPACES

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ABSTRACT. In this paper, we investigate the existence and diameter of the approximate fixed point results on E-metric spaces (not necessarily complete) by using various cyclic contraction mappings, including the B-cyclic contraction, the Bianchini cyclic contraction, the Hardy-Rogers cyclic contraction, and so on. Additionally, we prove the approximate fixed point results for rational type cyclic contraction mappings, which were discussed mainly in [35] and [37], in the setting of E-metric space. Also, a few examples are provided to demonstrate our findings. Subsequently, we discuss some applications of approximate fixed point results in the field of applied mathematics rigorously.

1. INTRODUCTION

Poincare [1], a French mathematician, developed the concept of fixed points by utilising operators in an abstract topological form while studying nonlinear equations in the early 1880s. The author, Liouville [2], created the sequential approximation method in 1837, and Picard introduced its logical technique in 1890. The renowned fixed point solution was authored in 1912 by Brouwer [3], and a lot of its subsequent findings were stated in a topological manner. Banach's Principle, in [4], is a powerful concept for locating fixed points, and it has been expanded and modified in different ways by several academics (see for example [5, 6, 7, 8, 9]). In the absence of exact fixed points, approximate fixed points may be used because the fixed point methods have overly strict limitations. An approximate fixed point is a point that is nearly located at its respective fixed point (with only ϵ - difference). Initially, Huang and Zhang [10]

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invented cone metric space, which vastly extends metric space, in 2007. They also discovered fixed point theorems for Banach-type [4], Kannan-type [11], and Chatterjea-type [12] contractions. Following that, a high proportion of fixed point outcomes in cone metric spaces was seen. Rawashdeh et al., in [13], established an E-metric space (also called ordered space), which is comparable to cone metric space, and demonstrated that the contraction sequence in E-metric spaces is a Cauchy sequence. One can see the recent work on E-metric space and its fixed point results in [10, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], and so on.

Definition 1.1. [13] *Let E be a real ordered vector space, E^+ be a non-empty closed and convex subset of E and 0_E be a zero element in E. Then E^+ is called a positive cone if it satisfies*

(i) *for all $p \in E^+$ and $k \geq 0$ imply $kp \in E^+$; and*
(ii) *for all $p \in E^+$ and $-p \in E^+$ imply $p = 0_E$.*

Definition 1.2. [13] *An ordered space E is a vector space over the real numbers with partial order relation $\preceq$ such that*

(i) *for all p, q and $r \in E, p \preceq q$ imply that $p + r \preceq q + r$; and*
(ii) *for all $k \in \mathbb{R}^+$ and $p \in E$ with $p \succeq 0_E$ imply $kp \succeq 0_E$.*

Definition 1.3. [13] *Let M be a non-empty set and let E be a real ordered vector space. An E-metric function $d : M \times M \to E$ such that for all p, q and $r \in M$, we have*

(i) *$d(p, q) \succeq 0_E$ and $d(p, q) = 0_E$ if and only if $p = q$;*
(ii) *$d(p, q) = d(q, p)$;*
(iii) *$d(p, q) \preceq d(p, r) + d(r, q)$.*

Then, the pair (M, d) is called an E-metric space.

Numerous authors, including Berinde, M., Berinde, V., and Mohsenialhosseini, S. A. M., etc. (refer to [25, 26, 27, 28, 29, 30, 31]), discussed and proved theorems concerning ϵ-fixed point locations using multiple operators on metric spaces, which is not necessarily complete. Moreover, an article [32] [Kirk. W. A., Srinivasan, P. S., Veeramani, P., *Fixed point for mappings satisfying cyclical contraction conditions*, Fixed point theory, 4, 2003, 79-89], long-windedly explains about the notion of cyclic mappings and its theorems. Further, the authors [33, 34, 35, 36, 37, 38] were used extended form of Banach fixed point theorem in the form of rational type contraction operators for finding new fixed point results. Additionally, in [39], the authors proved many fixed point results by using B-contraction operator.

This manuscript is laid out as follows: In Section 2, we recall the notations, basic notions, and essential definitions needed throughout the paper. In Section 3, we prove the main concept related to approximate fixed point results using both cyclic contraction and rational type cyclic contraction mappings. In Section 4, we go one step further and find the applications of approximate fixed point results in a wide range of mathematical topics. Finally, in Section 5, we reach a conclusion.

2. PRELIMINARIES

In this section, some notations, basic notions, essential definitions, and needed lemmas from earlier works are recalled. These are then employed throughout the remainder of the main results of the manuscript.

Definition 2.1. [25] *Let (M, d) be a metric space and $K : M \to M$, $\epsilon > 0, k \in M$. Then k is said to be an ϵ-fixed point (approximate fixed point) of K if*

$$d(k, Kk) < \epsilon.$$

In this paper we will denote the set of all ϵ-fixed point of K, for a given $\epsilon > 0$, by

$$F_{E\epsilon}(K) = \{k \in M : d(k, Kk) < \epsilon\}.$$

Definition 2.2. [25] *Let $K : M \to M$. Then K has an approximate fixed point property (a.f.p.p) if for every $\epsilon > 0$,*

$$F_{E\epsilon}(K) \neq \emptyset.$$

Lemma 2.3. [25] *Let U be a nonempty subset of a metric space (M, d) and $K : U \to U$ such that K is asymptotic regular, i.e., $d(K^n(k), K^{n+1}(k)) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $k \in U$. Then,*

$$F_{E\epsilon}(K) \neq 0, \text{ for all } \epsilon > 0.$$

Remark 2.4. [25] *In the following, by $\Delta(A)$ for a set $A \neq \emptyset$, we will understand the diameter of the set A, i.e.,*

$$\Delta(A) = sup\{d(k, l) : k, l \in A\}.$$

Definition 2.5. [25] *Let (M, d) be a metric space, $K : M \to M$ an operator and $\epsilon > 0$. We define the diameter of the set $F_{E\epsilon}(K)$, i.e.,*

$$\Delta(F_{E\epsilon}(K)) = sup\{d(k, l) : k, l \in F_{E\epsilon}(K)\}.$$

Theorem 2.6. [25] *Let A be a closed subset of a metric space (M, d) and $K : A \to M$ be a compact map. Then K has a fixed point if and only if it has an approximate fixed point property.*

Lemma 2.7. [25] *Let (M, d) be a metric space, $K : M \to M$ an operator and $\epsilon > 0$. We assume that:*

(i) *$F_{E\epsilon}(K) \neq \emptyset$; and*
(ii) *for every $\theta > 0$, there exists $\phi(\theta) > 0$ such that $d(k, l) - d(Kk, Kl) \leq \theta$ implies that $d(k, l) \leq \phi(\theta)$, for all $k, l \in F_{E\epsilon}(K)$.*

Then;

$$\Delta(F_{E\epsilon}(K)) \leq \phi(2\epsilon).$$

Definition 2.8. [32] *Let U and V be two nonempty subsets of a metric space (M, d). A mapping $K : U \cup V \to U \cup V$ is said to be a cyclic mapping if $K(U) \subseteq V$ and $K(V) \subseteq U$.*

Definition 2.9. [11] *Let U and V be two nonempty subsets of a metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is said to be a cyclic contraction if there exist $e \in [0, 1)$ such that*

$$d(Kk, Kl) \leq ed(k, l), \text{ for all } k \in U, l = Kk \in V.$$

Definition 2.10. [39] *Let U and V be two nonempty subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is said to be a B-cyclic contraction if there exists $e_1, e_2, e_3 \in (0, 1)$ with $e_1 + e_2 + e_3 < 1$ such that*

$$d(Kk, Kl) \leq e_1[d(k, Kk)+d(l, Kl)]+e_2d(k, l)+e_3[d(k, Kl)+d(l, Kk)], \text{ for all } k \in U, l \in V.$$

Definition 2.11. [40] *Let U and V be two nonempty subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is said to be a Ćirić-Reich-Rus cyclic contraction if there exists $e_1, e_2 \in (0, 1)$ with $e_1 + 2e_2 < 1$ such that*

$$d(Kk, Kl) \leq e_1d(k, l) + e_2(d(k, Kk) + d(l, Kl)), \text{ for all } k \in U, l = Kk \in V.$$

Definition 2.12. [5] *Let U and V be two nonempty subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is said to be a Bianchini cyclic contraction if there exist $e \in (0, 1)$ and for all $k \in U$, $l = Kk \in V$ such that $d(Kk, Kl) \leq eB(k, l)$, where $B(k, l) = \max \{d(k, Kk), d(l, Kl)\}$.*

Definition 2.13. [7] *Let U and V be two nonempty subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is said to be a Hardy-Rogers cyclic contraction if there exists $e_1, e_2, e_3, e_4, e_5 \in (0, 1)$ with $e_1 + e_2 + e_3 + e_4 + e_5 < 1$ such that*

$$d(Kk, Kl) \leq e_1d(k, l)+e_2d(k, Kk)+e_3d(l, Kl)+e_4d(k, Kl)+e_5d(l, Kk), \text{ for all } k \in U, l \in V.$$

3. Main Results

This section is split into a couple of parts. We present the approximate fixed point results for various types of cyclic contraction and rational type cyclic contraction mappings, respectively, in the following subsections.

3.1. **Results for various cyclic contraction mappings.** In this part, we prove the approximate fixed point theorems using various types of cyclic contraction mappings, including B-cyclic contraction, Bianchini cyclic contraction, Hardy-Rogers cyclic contraction, and their consequences on E-metric spaces (not necessarily complete).

Theorem 3.1. *Let U and V be two nonempty subsets of a E-metric space and a cyclic mapping $K : U \cup V \to U \cup V$ be a contraction. Then K has an ϵ-fixed point.*

Proof. Fix $k_0 \in U$ and $l = Kk_0 = k_1 \in V$. Define a sequence $\{k_n\}$ on $U \cup V$ such that $k_{n+1} = Kk_n$, for all $n \geq 0$. That is, $\{k_n\}$ is a Cauchy sequence. Which implies that, for all $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ implies $d(k_n, k_m) < \epsilon$. In particular, if $n \geq n_0, d(k_n, k_{n+1}) < \epsilon$. That is, $d(k_n, Kk_n) < \epsilon$. Therefore, $k_n \in F_{E\epsilon}(K) \neq \emptyset$, for all $\epsilon > 0$. Hence, K has an ϵ-fixed point. $\square$

Theorem 3.2. *Let U and V be two non-empty closed subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is a B-cyclic contraction. Then K has an ϵ-fixed point and*

$$\Delta(F_{E\epsilon}(K)) \leq \frac{(e_1 + e_3 + 1)2\epsilon}{1 - e_2 - 2e_3}, \text{ for all } \epsilon > 0.$$

Proof. Given K is B-cyclic contraction. Let $\epsilon > 0$, $k_0 \in U$ and $l = Kk_0 = k_1 \in V$. Define a sequence $\{k_n\}$ on $U \cup V$ such that $k_{n+1} = Kk_n,$ for all $n > 0$. Consider,

$$\begin{aligned}
d(K^n k, K^{n+1}k) &= d(K(K^{n-1}k), K(K^n k)) \\
&\leq e_1[d(K^{n-1}k, K^n k) + d(K^n k, K^{n+1}k)] + e_2 d(K^n k, K^{n-1}k) \\
&\quad + e_3[d(K^{n-1}k, K^{n+1}k) + d(K^n k, K^n k)] \\
&= e_1[d(K^{n-1}k, K^n k) + d(K^n k, K^{n+1}k)] + e_2 d(K^{n-1}k, K^n k) \\
&\quad + e_3[d(K^{n-1}k, K^n k) + d(K^n k, K^{n+1}k)] \\
&= \lambda d(K^{n-1}k, K^n k), \text{ where } \lambda = \frac{e_1 + e_2 + e_3}{1 - e_1 - e_3} \\
&\leq \lambda^2 d(K^{n-2}k, K^{n-1}k) \\
&\dots \\
&\leq \lambda^n d(k, Kk)
\end{aligned}$$

Since $d(K^n k, K^{n+1}k) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $k \in M$. That is, $\{k_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_{E\epsilon}(K) \neq \emptyset$. Therefore, K has an ϵ-fixed point. It means that, condition (i) of Lemma 2.7 is verified. To prove condition (ii) of Lemma 2.7. For that, fix on $\theta > 0$ and $k, l \in F_{E\epsilon}(K)$. Also, $d(k, l) - d(Kk, Kl) \leq \theta$. To claim $\phi(\epsilon) > 0$ exists.

$$\begin{aligned}
d(k, l) &\leq d(Kk, Kl) + \theta \\
&\leq e_1[d(k, Kk) + d(l, Kl)] + e_2 d(k, l) + e_3[d(k, Kl) + d(l, Kk)] + \theta \\
&= 2e_1\epsilon + e_2 d(k, l) + 2e_3\epsilon + 2e_3 d(k, l) + \theta \\
&= \left(\frac{2e_1\epsilon + 2e_3\epsilon + \theta}{1 - e_2 - 2e_3}\right) \\
&= \delta
\end{aligned}$$

So, for all $\theta > 0$, there exist $\phi(\theta) = \delta > 0$ such that $d(k, l) - d(Kk, Kl) \leq \theta \Rightarrow d(k, l) \leq \phi(\theta)$. By Lemma 2.7, $\Delta(F_{E\epsilon}(K)) \leq \phi(2\epsilon)$, for all $\epsilon > 0$. Hence,

$$\Delta(F_{E\epsilon}(K)) \leq \frac{(e_1 + e_3 + 1)2\epsilon}{1 - e_2 - 2e_3}, \text{ for all } \epsilon > 0.$$

□

Corollary 3.3. *Let U and V be a two non-empty closed subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is a cyclic contraction. Then K has an ϵ-fixed point and*

$$\Delta(F_{E\epsilon}(K)) \leq \frac{2\epsilon}{1-e_2}, \textit{ for all } \epsilon > 0.$$

Proof. Substituting $e_1 = e_3 = 0$ in Theorem 3.2 completes this corollary. □

Corollary 3.4. *Let U and V be a two non-empty closed subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is a Kannan cyclic contraction. Then K has an ϵ-fixed point and*

$$\Delta(F_{E\epsilon}(K)) \leq (1 + e_1)2\epsilon, \textit{ for all } \epsilon > 0.$$

Proof. Substituting $e_2 = e_3 = 0$ in Theorem 3.2 completes this corollary. □

Corollary 3.5. *Let U and V be a two non-empty closed subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is a Chatterjea cyclic contraction. Then K has an ϵ-fixed point and*

$$\Delta(F_{E\epsilon}(K)) \leq \frac{(e_3+1)2\epsilon}{1-2e_3}, \textit{ for all } \epsilon > 0.$$

Proof. Substituting $e_1 = e_2 = 0$ in Theorem 3.2 completes this corollary. □

Corollary 3.6. *Let U and V be a two non-empty closed subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is a Ćirić-Reich-Rus cyclic contraction. Then K has an ϵ-fixed point and*

$$\Delta(F_{E\epsilon}(K)) \leq \frac{(1+e_1)2\epsilon}{1-e_2}, \textit{ for all } \epsilon > 0.$$

Proof. Substituting $e_3 = 0$ in Theorem 3.2 completes this corollary. □

Theorem 3.7. *Let U and V be a two non-empty closed subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is a Bianchini cyclic contraction. Then K has an ϵ-fixed point*

$$\Delta(F_{E\epsilon}(K)) \leq (e + 2)\epsilon, \textit{ for all } \epsilon > 0.$$

Proof. Given K is Bianchini cyclic contraction. Let $\epsilon > 0$, $k_0 \in U$ and $l = Kk_0 = k_1 \in V$. Define a sequence $\{k_n\}$ on $U \cup V$ such that $k_{n+1} = Kk_n$, for all $n > 0$.

Case 1. *If $B(k, l) = d(k, Kk)$. Then, Definition 2.12 becomes:*

$$\begin{aligned}
d(Kk, Kl) &\leq ed(k, Kk)\\
\textit{Substitute } l = Kk,\ d(Kk, K^2k) &\leq ed(k, Kk)\\
\textit{Again substitute } k = Kk,\ d(K^2k, K^3k) &\leq ed(Kk, K^2k)\\
&= e^2 d(k, Kk)\\
&\vdots\\
d(K^nk, K^{n+1}k) &\leq e^n d(k, Kk).
\end{aligned}$$

Case 2. *If $B(k,l) = d(l, Kl)$. Then, Definition 2.12 becomes:*

$$d(Kk, Kl) \leq ed(l, Kl)$$
$$\textit{Substitute } l = Kk,\ d(Kk, K^2k) \leq ed(Kk, K^2k)$$

This is impossible because $e \in (0,1)$. Therefore, **Case 2** does not exist. Now by **Case 1**, $d(K^n k, K^{n+1}k) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $k \in M$. That is, $\{k_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_{E\epsilon}(K) \neq \emptyset$. Therefore, K has an ϵ-fixed point. It means that, condition (i) of Lemma 2.7 is verified. To prove condition (ii) of Lemma 2.7. For that, fix on $\theta > 0$ and $k, l \in F_{E\epsilon}(K)$. Also, $d(k,l) - d(Kk, Kl) \leq \theta$. To claim $\phi(\epsilon) > 0$ exists.

$$\begin{aligned} d(k,l) &\leq d(Kk, Kl) + \theta \\ &\leq eB(k,l) + \theta \\ &= ed(k, Kk) + \theta \\ &= e\epsilon + \theta \\ &= \delta \end{aligned}$$

So, for all $\theta > 0$, there exist $\phi(\theta) = \delta > 0$ such that $d(k,l) - d(Kk, Kl) \leq \theta \Rightarrow d(k,l) \leq \phi(\theta)$. By Lemma 2.7 , $\Delta(F_{E\epsilon}(K)) \leq \phi(2\epsilon)$, for all $\epsilon > 0$. Hence,

$$\Delta(F_{E\epsilon}(K)) \leq (e+2)\epsilon, \text{ for all } \epsilon > 0.$$

□

Example 3.8. *Let (M, d) be a E-metric space and $M = \mathbb{R}$. If $U = V = [0,1]$ and a self mapping $K : [0,1] \to [0,1]$ defined by*

$$Kk = \begin{cases} \dfrac{1}{4} & \textit{when } k = 1 \\[2ex] \dfrac{1}{2} & \textit{when } k \in [0,1) \end{cases}$$

If $k = 15/16$ and $l = 1$, then K satisfies the definition 2.12.

Example 3.9. *Let (M, d) be a E-metric space and $M = \mathbb{R}$. If $U = V = [0,1]$ and a self mapping $K : [0,1] \to [0,1]$ defined by*

$$Kk = \begin{cases} \dfrac{8}{14} & \textit{when } k \in [0,1/2] \\[2ex] \dfrac{4}{14} & \textit{when } k \in (1/2,1] \end{cases}$$

If $k = 1/4$ and $l = 14/15$, then K satisfies the definition 2.12.

Corollary 3.10. *Let U and V be two nonempty closed subsets of a E-metric space (M, d). A cyclic map $K : U \cup V \to U \cup V$ satisfies $d(Kk, Kl) \leq ed(Kk, k)$ for all $k \in U, l = Kk \in V$ and for some $e \in (0, 1)$. Then K has an ϵ-fixed point and*

$$\Delta(F_{E\epsilon}(K)) \leq (e + 2)\epsilon, \text{ for all } \epsilon > 0.$$

Proof. Substituting $B(k, l) = d(Kk, k)$ in Theorem 3.7 completes this corollary. □

Theorem 3.11. *Let U and V be two nonempty closed subsets of a E-metric space (M, d). A cyclic map $K : U \cup V \to U \cup V$ is a Hardy-Rogers cyclic contraction. Then K has an ϵ-fixed point and*

$$\Delta(F_{E\epsilon}(K)) \leq \frac{(e_2 + e_3 + e_4 + e_5 + 2)\epsilon}{1 - e_1 - e_4 - e_5}, \text{ for all } \epsilon > 0.$$

Proof. Given K is Hardy-Rogers cyclic contraction. Let $\epsilon > 0$, $k_0 \in U$ and $l = Kk_0 = k_1 \in V$. Define a sequence $\{k_n\}$ on $U \cup V$ such that $k_{n+1} = Kk_n$, for all $n \geq 0$. Consider,

$$\begin{aligned} d(K^n k, K^{n+1} k) &= d(K(K^{n-1}k), K(K^n k)) \\ &\leq k_1 d(K^{n-1}k, K^n k) + k_2 d(K^{n-1}k, K^n k) + k_3 d(K^n k, K^{n+1}k) \\ &\quad + k_4 d(K^{n-1}k, K^{n+1}k) + k_5 d(K^n k, K^n k) \\ &\leq \lambda d(K^{n-1}k, K^n k), \text{ where } \lambda = \frac{e_1 + e_2 + e_4}{1 - e_1 - e_4} \\ &\leq \lambda^2 d(K^{n-2}k, K^{n-1}k) \\ &\cdots \\ &\leq \lambda^n d(k, Kk) \end{aligned}$$

Since $d(K^n k, K^{n+1}k) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $k \in M$. That is, $\{k_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_{E\epsilon}(K) \neq \emptyset$. Therefore, K has an ϵ-fixed point. It means that, condition (i) of Lemma 2.7 is verified. To prove condition (ii) of Lemma 2.7. For that, fix on $\theta > 0$ and $k, l \in F_{E\epsilon}(K)$. Also, $d(k, l) - d(Kk, Kl) \leq \theta$. To claim $\phi(\epsilon) > 0$ exists.

$$\begin{aligned} d(k, l) &\leq d(Kk, Kl) + \theta \\ &\leq e_1 d(k, l) + e_2 d(k, Kk) + e_3 d(l, Kl) + e_4 d(k, Kl) + e_5 d(l, Kk) + \theta \\ &= \left(\frac{e_2\epsilon + e_3\epsilon + e_4\epsilon + e_5\epsilon + \theta}{1 - e_1 - e_4 - e_5} \right) \\ &= \delta \end{aligned}$$

So, for all $\theta > 0$, there exist $\phi(\theta) = \delta > 0$ such that $d(k, l) - d(Kk, Kl) \leq \theta \Rightarrow d(k, l) \leq \phi(\theta)$. By Lemma 2.7, $\Delta(F_{E\epsilon}(K)) \leq \phi(2\epsilon)$, for all $\epsilon > 0$. Hence,

$$\Delta(F_{E\epsilon}(K)) \leq \frac{(e_2 + e_3 + e_4 + e_5 + 2)\epsilon}{1 - e_1 - e_4 - e_5}, \text{ for all } \epsilon > 0.$$

□

Corollary 3.12. *Let U and V be a two non-empty closed subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is a Ćirić cyclic contraction. Then K has an ϵ-fixed point and*

$$\Delta(F_{E\epsilon}(K)) \leq \frac{(e_2 + e_3 + 2e_4 + 2)\epsilon}{1 - e_1 - 2e_4}, \text{ for all } \epsilon > 0.$$

Proof. Substituting $e_5 = e_4$ in Theorem 3.11 completes this corollary. □

Corollary 3.13. *Let U and V be a two non-empty closed subsets of a E-metric space (M, d). A cyclic mapping $K : U \cup V \to U \cup V$ is a Reich cyclic contraction. Then K has an ϵ-fixed point and*

$$\Delta(F_{E\epsilon}(K)) \leq \frac{(e_2 + e_3 + 2)\epsilon}{1 - e_1}, \text{ for all } \epsilon > 0.$$

Proof. Substituting $e_4 = e_5 = 0$ in Theorem 3.11 completes this corollary. □

3.2. **Results for various rational type cyclic contraction mappings.** In this part, we prove some approximate fixed point theorems using various rational cyclic contraction mappings on E-metric spaces (not necessarily complete). These rational contractions were discussed mainly in [35] and [37].

Theorem 3.14. *Let U and V be two nonempty closed subsets of a E-metric space (M, d) and $K : U \cup V \to U \cup V$ be a cyclic mapping. Then there exist $e \in [0, 1)$ and $d(k, Kl) + d(l, Kk) \neq 0$ such that*

$$d(Kk, Kl) \leq \frac{e[d(k, Kk)d(k, Kl) + d(l, Kl)d(l, Kk)]}{d(k, Kl) + d(l, Kk)},$$

for all $k \in U$ and $l = Kk \in V$. Then K has an ϵ-fixed point and

$$\Delta(F_{E\epsilon}(K)) < \frac{(e^2 + 6e + 9)\epsilon^2 + (e + 1)\epsilon}{2}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0, k_0 \in U \cup V$. Define a sequence $\{k_n\}$ such that $k_{n+1} = Kk_n$, for all $n \geq 0$. Consider,

$$\begin{aligned}
d(K^n k, K^{n+1} k) &= d(K(K^{n-1}k), K(K^n k)) \\
&\leq e\left[\frac{d(K^{n-1}k, K^n k)d(K^{n-1}k, K^{n+1}k) + d(K^n k, K^{n+1}k)d(K^n k, K^n k)}{d(K^{n-1}k, K^{n+1}k) + d(K^n k, K^n k)}\right] \\
&\leq ed(K^{n-1}k, K^n k) \\
&\leq e^2 d(K^{n-2}k, K^{n-1}k) \\
&\dots \\
&\leq e^n d(k, Kk)
\end{aligned}$$

Since $d(K^n k, K^{n+1} k) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $k \in M$. This implies that, $\{k_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_{E\epsilon}(K) \neq \emptyset$. That is, K has an ϵ-fixed point. It means that, condition (i) of Lemma 2.7 is verified. To prove condition (ii) of Lemma 2.7. For that, fix on $\theta > 0$ and $k, l \in F_{E\epsilon}(K)$. Also, $d(k,l) - d(Kk, Kl) \leq \theta$. To claim $\phi(\epsilon) > 0$ exists.

$$\begin{aligned} d(k,l) &\leq e\left[\frac{d(k,Kk)d(k,Kl) + d(l,Kl)d(l,Kk)}{d(k,Kl) + d(l,Kk)}\right] + 2\epsilon \\ &= e\left[\frac{\epsilon[d(k,l)+\epsilon] + \epsilon[d(k,l)+\epsilon]}{2d(k,l)+2\epsilon}\right] + 2\epsilon \\ &= \frac{2e\epsilon d(k,l) + 2e\epsilon^2 + 4\epsilon d(k,l) + 4\epsilon^2}{2d(k,l)+2\epsilon} \end{aligned}$$

On simplyfying, we get

$$2[d(k,l)]^2 \leq 2\epsilon(1+e)d(k,l) + 2\epsilon^2(e+2)$$

Which implies that $a = 2, b = -2\epsilon(1+e)$ and $c = -2\epsilon^2(e+2)$. Therefore,

$$\begin{aligned} d(k,l) &\leq \frac{2\epsilon(1+e) \pm \sqrt{4\epsilon^2(1+e)^2 + 16\epsilon^2(e+2)}}{4} \\ &= \frac{2\epsilon(1+e) + \sqrt{4\epsilon^2(1+2e+e^2) + 16\epsilon^2 e + 32\epsilon^2}}{4} \\ &= \frac{2\epsilon(1+e) + \sqrt{4\epsilon^2 + 8\epsilon^2 e + 4e^2\epsilon^2 + 16\epsilon^2 e + 32\epsilon^2}}{4} \\ &= \frac{2\epsilon(1+e) + \sqrt{36\epsilon^2 + 24\epsilon^2 e + 4e^2\epsilon^2}}{4} \\ &= \frac{\epsilon(1+e) + \sqrt{9\epsilon^2 + 6\epsilon^2 e + e^2\epsilon^2}}{2} \\ &< \frac{\epsilon + \epsilon e + 9\epsilon^2 + 6\epsilon^2 e + e^2\epsilon^2}{2} \end{aligned}$$

Hence,

$$\Delta(F_{E\epsilon}(K)) < \frac{(e^2+6e+9)\epsilon^2 + (e+1)\epsilon}{2}, \text{ for all } \epsilon > 0.$$

$\square$

Example 3.15. *Let (M,d) be a E-metric space. Let $M = (0, 1/2]$ be endowed with usual metric. Let $K : M \to M$ be defined by $Kk = k/2$, for all $k \in M$. To show that the Theorem 3.13 is satisfied. For that choose $e = 1/4$ and $\epsilon = 1/2$. Also $k = 1/2$, $l = 1/4 \in M$. Then, $d(k,Kk) = 1/3 < 1/2; d(l,Kl) = 1/8 < 1/2$. We have $d(k,l) = 1/4$ and*

$$\Delta(F_{E\epsilon}(K)) < \frac{\epsilon^2(e^2+6e+9) + \epsilon(e+1)}{2} = \frac{193}{128}.$$

Theorem 3.16. *Let U and V be two nonempty closed subsets of a E-metric space (M,d) and $K : U \cup V \to U \cup V$ be a cyclic mapping. Then there exists $e_1, e_2 \in [0, 1/2)$ and $k \neq l$ such* *that*

$$d(Kk,Kl) \leq \frac{e_1 d(k,Kk)d(l,Kl)}{d(k,l)} + e_2 d(k,l),$$

for all $k \in U$ and $l = Kk \in V$. Then K has an ϵ-fixed point and

$$\Delta(F_{E\epsilon}(K)) < \epsilon\left(\frac{2}{1-e_2} + e_1\right), \textit{ for all } \epsilon > 0.$$

Proof. Using the same procedure as in Theorem 3.14, we obtain $F_{E\epsilon}(K) \neq \emptyset$. That is, K has an ϵ-fixed point. It means that, condition (i) of Lemma 2.7 is verified. To prove condition (ii) of Lemma 2.7. For that, fix on $\theta > 0$ and $k, l \in F_{E\epsilon}(K)$. Also, $d(k,l) - d(Kk,Kl) \leq \theta$. To claim $\phi(\epsilon) > 0$ exists. Consider,

$$\begin{aligned}
d(k,l) &\leq d(Kk,Kl) + \theta \\
&\leq \frac{e_1 d(k,Kk)d(l,Kl)}{d(k,l)} + e_2 d(k,l) + 2\epsilon \\
(1-e_2)d(k,l)^2 &\leq e_1\epsilon^2 + 2\epsilon d(k,l) \\
\left[d(k,l) - \left(\frac{\epsilon}{1-e_2}\right)\right]^2 &\leq \frac{e_1\epsilon^2}{1-e_2} + \left(\frac{\epsilon}{1-e_2}\right)^2 \\
d(k,l) - \left(\frac{\epsilon}{1-e_2}\right) &\leq \sqrt{\frac{(1-e_2)e_1\epsilon^2 + \epsilon^2}{(1-e_2)^2}} \\
d(k,l) &= \sqrt{\frac{e_1\epsilon^2 - e_1e_2\epsilon^2 + \epsilon^2}{(1-e_2)^2}} + \left(\frac{\epsilon}{1-e_2}\right) \\
&= \sqrt{\frac{e_1\epsilon^2 + (1-e_1e_2)\epsilon^2}{(1-e_2)^2}} + \left(\frac{\epsilon}{1-e_2}\right) \\
&= \left(\frac{\epsilon}{1-e_2}\right)\sqrt{e_1 + 1 - e_1e_2} + \left(\frac{\epsilon}{1-e_2}\right) \\
&< \left(\frac{\epsilon}{1-e_2}\right)(1 + e_1 + 1 - e_1e_2) \\
d(k,l) &= \left(\frac{\epsilon}{1-e_2}\right)(2 + e_1(1-e_2))
\end{aligned}$$

Hence,

$$\Delta(F_{E\epsilon}(K)) < \epsilon\left(\frac{2}{1-e_2} + e_1\right), \text{ for all } \epsilon > 0.$$

$\square$

Theorem 3.17. *Let U and V be two nonempty closed subsets of a E-metric space (M,d) and $K: U\cup V \to U\cup V$ be a cyclic mapping. Then, there exists $e_1, e_2 \in [0,1/2)$ such that*

$$d(Kk,Kl) \le \frac{e_1 d(l,Kl)[1+d(k,Kk)]}{1+d(k,l)} + e_2 d(k,l),$$

for all $k\in U$, $l = Kk \in V$. Then K has an ϵ-fixed point and

$$\Delta(F_{E\epsilon}(K)) < \frac{e_2^2 - e_2 + 6\epsilon + 4\epsilon^2(1+e_1-e_1e_2) + 4\epsilon(e_1-e_2-e_1e_2)}{2(1-e_2)}, \textit{ for all } \epsilon > 0.$$

Proof. Using the same procedure as in Theorem 3.14, $F_{E\epsilon}(K) \neq \emptyset$. That is, K has an ϵ-fixed point. It means that, condition (i) of Lemma 2.7 is verified. To prove condition (ii) of Lemma 2.7. For that, fix on $\theta > 0$ and $k,l \in F_{E\epsilon}(K)$. Also, $d(k,l) - d(Kk,Kl) \le \theta$. To claim $\phi(\epsilon) > 0$ exists. Consider,

$$\begin{aligned} d(k,l) &\le d(Kk,Kl) + \theta \\ &\le \frac{e_1 d(l,Kl)[1+d(k,Kk)]}{1+d(k,l)} + k_2 d(k,l) + 2\epsilon \\ &\le \frac{e_1\epsilon[1+\epsilon]}{1+d(k,l)} + e_2 d(k,l) + 2\epsilon \end{aligned}$$

Which implies,

$$[d(k,l)]^2 + \left(\frac{1-e_2-2\epsilon}{1-e_2}\right) d(k,l) \le \frac{e_1\epsilon^2 + e_1\epsilon + 2\epsilon}{1-e_2}$$

Taking square,

$$\left[d(k,l) + \frac{1-e_2-2\epsilon}{2(1-e_2)}\right]^2 \le \left(\frac{1-e_2-2\epsilon}{2(1-e_2)}\right)^2 + \frac{e_1\epsilon^2 + e_1\epsilon + 2\epsilon}{1-e_2}$$

On simplifying,

$$\begin{aligned} &d(k,l) \\ &\le \frac{1}{2(1-e_2)}\sqrt{e_2^2 - 2e_2 + 4\epsilon^2(1+e_1-e_1e_2) + 4\epsilon(1+e_1-e_2-e_1e_2) + 1} + \left(\frac{e_2+2\epsilon-1}{2(1-e_2)}\right) \\ &< \frac{1}{2(1-e_2)}\left(e_2^2 - 2e_2 + 4\epsilon^2(1+e_1-e_1e_2) + 4\epsilon(1+e_1-e_2-e_1e_2) + 1 + e_2 + 2\epsilon - 1\right) \end{aligned}$$

That is,

$$\begin{aligned} d(k,l) &< \frac{e_2^2 - 2e_2 + e_2 + 2\epsilon + 4\epsilon^2(1+e_1-e_1e_2) + 4\epsilon(1+e_1-e_2-e_1e_2)}{2(1-e_2)} \\ &= \frac{e_2^2 - e_2 + 6\epsilon + 4\epsilon^2(1+e_1-e_1e_2) + 4\epsilon(e_1-e_2-e_1e_2)}{2(1-e_2)} \end{aligned}$$

Hence,

$$\Delta(F_{E\epsilon}(K)) < \frac{e_2^2 - e_2 + 6\epsilon + 4\epsilon^2(1 + e_1 - e_1e_2) + 4\epsilon(e_1 - e_2 - e_1e_2)}{2(1 - e_2)}, \text{ for all } \epsilon > 0.$$

□

Example 3.18. *Let (M, d) be a E-metric space. Let $M = (0, 1/2]$ be endowed with usual metric. Let $K : M \to M$ be defined by $Kk = k/2$, for all $k \in M$. For that, choose $e_1 = 1/4, e_2 = 1/5$ and $\epsilon = 1/2$. Also $k = 1/2, l = 1/4 \in M$. Then, $d(k, Kk) = 1/3 < 1/2$, $d(l, Kl) = 1/8 < 1/2$. We have $d(k, l) = 1/4$ and*

$$\frac{e_2^2 - e_2 + 6\epsilon + 4\epsilon^2(1 + e_1 - e_1e_2) + 4\epsilon(e_1 - e_2 - e_1e_2)}{2(1 - e_2)} \leq \frac{207}{80}.$$

Theorem 3.19. *Let U and V be two nonempty closed subsets of a E-metric space (M, d) and $K : U \cup V \to U \cup V$ be a cyclic mapping. Then, there exist $e_1, e_2 \in [0, 1/2)$ with $e_1 + e_2 < 1$, and $d(l, Kl) + d(k, l) > 0$ such that*

$$d(Kk, Kl) \leq \frac{e_1[d(k, Kk)d(k, Kl)d(l, Kl)]}{d(l, Kl) + d(k, l)} + e_2d(k, l),$$

for all $k \in U$, $l = Kk \in V$. Then K has an ϵ-fixed point and

$$\Delta(F_{E\epsilon}(K)) < \frac{1}{2(1 - e_2)}[\epsilon^4e_1^2 + \epsilon^3(6e_1 - 2e_1e_2) + \epsilon^2(e_1 + 10) + \epsilon(1 + e_2)], \textit{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$, $k_0 \in U \cup V$. Define a sequence $\{k_n\}$ such that $k_{n+1} = Kk_n$, for all $n \geq 0$. Consider,

$$d(K^{n+1}k, K^nk) = d(K(K^nk), K(K^{n-1}k))$$

and using the same procedure as in Theorem 3.14, $F_{E\epsilon}(K) \neq \emptyset$. That is, K has an ϵ-fixed point. It means that, condition (i) of Lemma 2.7 is verified. To prove condition (ii) of Lemma 2.7. For that, fix on $\theta > 0$ and $k, l \in F_{E\epsilon}(K)$. Also, $d(k, l) - d(Kk, Kl) \leq \theta$. To claim $\phi(\epsilon) > 0$ exist.

$$\begin{aligned} d(k, l) &\leq d(Kk, Kl) + \theta \\ &\leq \frac{e_1[d(k, Kk)d(k, Kl)d(l, Kl)]}{d(l, Kl) + d(k, l)} + e_2d(k, l) + 2\epsilon. \end{aligned}$$

Substituting the ϵ value, we get

$$d(k, l) \leq \frac{[\epsilon e_1d(k, l) + e_1\epsilon^2]\epsilon}{d(k, l) + \epsilon} + e_2d(k, l) + 2\epsilon$$

Which implies,

$$(1 - e_2)d(k, l) \leq \frac{e_1\epsilon^2d(k, l) + \epsilon^3e_1}{d(k, l) + \epsilon} + 2\epsilon$$

On simplifying,

$$[d(k,l)]^2 + \frac{d(k,l)[-e_2\epsilon - \epsilon - \epsilon^2 e_1]}{1-e_2} \leq \frac{2\epsilon^2 + e_1\epsilon^3}{1-e_2}$$

Taking square,

$$\left[d(k,l) + \frac{-e_2\epsilon - \epsilon - \epsilon^2 e_1}{2(1-e_2)}\right]^2 \leq \frac{2\epsilon^2 + e_1\epsilon^3}{1-e_2} + \left[\frac{-e_2\epsilon - \epsilon - \epsilon^2 e_1}{2(1-e_2)}\right]^2$$

Which implies that,

$$\begin{aligned}
d(k,l) &\leq \frac{e_2\epsilon + \epsilon + \epsilon^2 e_1}{2(1-e_2)} + \sqrt{\frac{2\epsilon^2 + e_1\epsilon^3}{1-e_2} + \left[\frac{-e_2\epsilon - \epsilon - \epsilon^2 e_1}{2(1-e_2)}\right]^2} \\
&= \frac{e_2\epsilon + \epsilon + \epsilon^2 e_1}{2(1-e_2)} + \sqrt{\frac{4(1-e_2)(2\epsilon^2 + e_1\epsilon^3) + (-e_2\epsilon - \epsilon - \epsilon^2 e_1)^2}{4(1-e_2)^2}} \\
&= \frac{1}{2(1-e_2)}\left[e_2\epsilon + \epsilon + \epsilon^2 e_1 + \sqrt{4(1-e_2)(2\epsilon^2 + e_1\epsilon^3) + (-e_2\epsilon - \epsilon - \epsilon^2 e_1)^2}\right] \\
&< \frac{1}{2(1-e_2)}\left[e_2\epsilon + \epsilon + \epsilon^2 e_1 + 4(1-e_2)(2\epsilon^2 + e_1\epsilon^3) + (-e_2\epsilon - \epsilon - \epsilon^2 e_1)^2\right]
\end{aligned}$$

Hence,

$$\Delta(F_{E\epsilon}(K)) < \frac{1}{2(1-e_2)}[\epsilon^4 e_1^2 + \epsilon^3(6e_1 - 2e_1e_2) + \epsilon^2(e_1 + 10) + \epsilon(1 + e_2)], \text{ for all } \epsilon > 0.$$

$\square$

Theorem 3.20. *Let U and V be two nonempty closed subsets of a E-metric space (M, d) and $K : U \cup V \to U \cup V$ be a cyclic mapping. Then there exists $e_1, e_2 \in (0, 1)$ with $e_1 + e_2 < 1$ and $d(l, Kl) + d(k, l) > 0$ such that*

$$d(Kk, Kl) \leq \frac{e[d(k, Kk)d(k, Kl)d(l, Kl)]}{d(l, Kl) + d(k, l)} + \varnothing d(k, l),$$

for all $k \in U$, $l = Kk \in V$. Then K has an ϵ-fixed point and $\Delta(F_{E\epsilon}(K))$ is imperfect.

Proof. Using the same procedure as in Theorem 3.14, $F_{E\epsilon}(K) \neq \emptyset$. That is, K has an ϵ-fixed point. It means that, condition (i) of Lemma 2.7 is verified. To prove condition (ii) of Lemma 2.7. For that, fix on $\theta > 0$ and $k, l \in F_{E\epsilon}(K)$. Also, $d(k, l) - d(Kk, Kl) \leq \theta$. To claim $\phi(\epsilon) > 0$ exists. Consider,

$$\begin{aligned}
d(k,l) &\leq [d(Kk, Kl)] + \theta \\
&\leq \frac{k[d(k, Kk)d(k, Kl)d(l, Kl)]}{d(l, Kl) + d(k, l)} + \varnothing d(k, l) + 2\epsilon \\
d(k,l) &\leq \frac{[d(k,l) + \epsilon]k\epsilon^2}{\epsilon + d(k,l)} + \varnothing d(k, l) + 2\epsilon
\end{aligned}$$

Since $\varnothing(p) < p$, for every $p > 0$, we have

$$d(k,l) \leq \frac{[d(k,l)+\epsilon]k\epsilon^2}{\epsilon + d(k,l)} + d(k,l) + 2\epsilon$$

On simplifying,

$$\begin{aligned} \frac{-[d(k,l)+\epsilon]k\epsilon^2}{\epsilon + d(k,l)} &\leq 2\epsilon \\ -[d(k,l)+\epsilon]k\epsilon^2 &\leq 2\epsilon^2 + 2d(k,l)\epsilon \\ (-k\epsilon^2 - 2\epsilon)d(k,l) &\leq 2\epsilon^2 + k\epsilon^3 \\ (-k\epsilon - 2)d(k,l) &\leq 2\epsilon + k\epsilon^2 \\ (k\epsilon + 2)d(k,l) &\geq -2\epsilon - k\epsilon^2 \\ \Delta(F_{E\epsilon}(K)) &\geq \frac{-(k\epsilon^2 + 2\epsilon)}{2 + k\epsilon} \end{aligned}$$

Which implies $\Delta(F_{E\epsilon}(K))$ is imperfect. □

4. Applications

Approximate fixed point theory covers a wide range of applications in applied mathematics, particularly differential geometry, numerical analysis, and so on. By reading [41, 42] and the references therein, one can find a variety of applications involving approximate fixed point results in the field of mathematics. The examples below demonstrate how to apply approximate fixed point findings to differential equations.

Example 4.1. *Let us consider* $l''(k) = 6l^2(k), 0 \leq k \leq 1$ *subect to* $l(0) = 1/4$ *and* $l(1) = 1/9$. *Here, the exact solution is* $l_0(k) = -5k/36 + 1/4$. *Consider a mapping* $K : [0,1] \to [0,1]$ *is defined by*

$$\begin{aligned} K(l) &= l + \int_0^1 G(k,s)[l''(s) - \phi(s, l(s), l'(s))]ds \\ &= \frac{-5k}{36} + \frac{1}{4} - \int_0^1 G(k,s)\phi(s, l(s), l'(s))ds \\ &= \frac{-5k}{36} + \frac{1}{4} - \int_0^1 G(k,s)6l''(s)ds \end{aligned}$$

Consider,

$$\begin{aligned}
|K(l_1)-K(l_2)| &= 6\left|-\int_0^1 G(k,s)l_1^2(s)ds+\int_0^1 G(k,s)l_2^2(s)ds\right| \\
&= 6\left(\int_0^1 |G(k,s)|^2 ds\right)^{\frac{1}{2}}\left(\int_0^1 |l_2^2(s)-l_1^2(s)|^2 ds\right)^{\frac{1}{2}} \\
&\le \frac{1}{4\sqrt{3}}\left(\int_0^1 |l_2^2(s)-l_1^2(s)|^2 ds\right)^{\frac{1}{2}} \\
&< \sup_{[0,1]}|l_1(s)-l_2(s)|
\end{aligned}$$

Hence, K is a contraction. So, by Theorem 3.1, it has an ϵ-fixed point.

Example 4.2. *Let us consider $l''(s)=3l^2(s)/2, 0\le k\le 1$ subect to $l(0)=4$ and $l(1)=1$. Here, the exact solution is $l(s)=4/(1+s)^2$. Consider a mapping $K:\ [0,1]\to[0,1]$ by*

$$K(l)=l+\int_0^1 G(k,s)[l''(s)-\phi(s,l(s))]ds \tag{4.1}$$

Consider, $l''(k)=0$ which implies

$$l(k)=c_1k+c_2 \tag{4.2}$$

By using the initial conditions, we have $c_2=4$ and $c_1=-3$. Then (4.2) becomes $l(k)=-3k+4$. From (4.1), we get

$$\begin{aligned}
K(l) &= -3k+4+\int_0^1 G(k,s)[l''(s)-\phi(s,l(s))]ds \\
&= -3k+4+\int_0^1 G(k,s)l''(s)ds-\int_0^1 G(k,s)\phi(s,l(s))ds \\
&= -3k+4+\int_0^1 G(k,s)\frac{3}{2}l^2(s)ds
\end{aligned}$$

Consider,

$$
\begin{aligned}
|K(l_1) - K(l_2)| &= \left| -\int_0^1 G(k,s)\frac{3}{2}l_1^2(s)ds + \int_0^1 G(k,s)\frac{3}{2}l_2^2(s)ds \right| \\
&= \frac{3}{2}\left| \int_0^1 G(k,s)[l_2^2(s) - l_1^2(s)]ds \right| \\
&\le \frac{3}{2}\left(\int_0^1 |G(k,s)|^2 ds \right)^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
&\le \frac{3}{2}\left(\int_0^k s^2(1-k)^2 ds + \int_k^1 k^2(1-s)^2 ds \right)^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
&\le \frac{3}{2}\left\{ \frac{(1-k)^2k^3}{3} + \frac{k^2(1-k)^3}{3} \right\}^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
&\le \frac{3}{2}\left\{ \frac{(1-k)^2}{3}[k^3 + k^2(1-k)] \right\}^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
&\le \frac{3}{2}\left\{ \frac{(1-k)^2k^2}{3} \right\}^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
&\le \frac{3}{8\sqrt{3}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
&= \frac{\sqrt{3}}{8} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
&\le \frac{\sqrt{3}}{8} \sup_{[0,1]} |l_2(s) - l_1(s)| \\
&\le \sup_{[0,1]} |l_2(s) - l_1(s)|
\end{aligned}
$$

Hence, K is a contraction. So, by Theorem 3.1, it has an ϵ-fixed point.

Remark 4.3. *In the above section, we have proved many approximate fixed point results by using various cyclic contraction mappings in E-metric space (not necessarily complete). The following table shows the diameters of various cyclic contraction operators and the diameters of a few rational type cyclic contraction operators.*

S. No	*Operator(s)*	*Diameter, for every* $\epsilon > 0$, $\Delta(F_{E\epsilon}(K)$
1	*Contraction* [11]	$\leq \dfrac{2\epsilon}{1-e_2}$
2	*Kannan* [11]	$\leq (l+e_1)2\epsilon$
3	*Chatterjea* [12]	$\leq \dfrac{(e_3+1)2\epsilon}{1-2e_3}$
4	B*-contraction* [39]	$\leq \dfrac{(e_1+e_3+1)2\epsilon}{1-e_2-2e_3}$
5	*Bianchini* [5]	$\leq (e+2)\epsilon$
6	*Hardy-Rogers* [7]	$\leq \dfrac{(e_2+e_3+e_4+e_5+2)\epsilon}{1-e_1-e_4-e_5}$
7	*Ćirić* [6]	$\leq \dfrac{(e_2+e_3+2e_4+2)\epsilon}{1-e_1-2e_4}$
8	*Ćirić-Reich-Rus* [40]	$\leq \dfrac{(1+e_1)2\epsilon}{1-e_1}$
9	*Reich* [8]	$\leq \dfrac{(e_2+e_3+2)\epsilon}{1-e_1}$
10	*Zamfirescu* [9]	$\leq \dfrac{(1+\delta)2\epsilon}{1-\delta}$
11	*Mohseni-saheli* [30]	$\leq \dfrac{(1+e)2\epsilon}{1-2e}$
12	*Mohseni-semi* [30]	$\leq \dfrac{(e+2)\epsilon}{1-e}$
13	*Weak contraction* [25]	$\leq \dfrac{(2+W)\epsilon}{1-e-W}$
14	*Contraction* (3.1)	$< \dfrac{(e^2+6e+9)\epsilon^2+(e+1)\epsilon}{2}$
15	*Contraction* (3.2)	$< \left(\dfrac{2}{1-e_2}+e_1\right)\epsilon$
16	*Contraction* (3.3)	$< \dfrac{6\epsilon+4\epsilon^2(1+e_1-e_1e_2)+4\epsilon(e_1-e_2-e_1e_2)}{2(1-e_2)}$
17	*Contraction* (3.4)	$< \dfrac{\epsilon^3(6e_1-2e_1e_2)+\epsilon^2(e_1+10)+\epsilon(1+e_2)}{2(1-e_2)}$
18	*Contraction* (3.5)	$Imperfect$

5. CONCLUSION

In this paper, some approximate fixed point theorems are established in E-metric space by utilizing various types of cyclic contraction mappings. Further, some approximate fixed point theorems are newly developed for rational type cyclic contraction mappings in the setting of E-metric space. It is worth observing that in the limiting case $\epsilon \longrightarrow 0$, all the results established in the present paper produces more restricted approximate fixed points. Approximate fixed points are consequently not less important than fixed points. As various future results can be demonstrated in a smaller setting to ensure the existence of the approximate fixed points.

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CONFLICT OF INTEREST

All the authors declare that they have no competing interest.

AUTHOR CONTRIBUTIONS

All authors contributed equally, read and approved the final manuscript.

DATA AVAILABILITY

No data was used for the research described in the article.

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