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APPROXIMATION FORMULAS FOR SHORT-MATURITY NEAR-THE-MONEY IMPLIED VOLATILITIES IN THE HESTON AND SABR MODELS

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ABSTRACT. Approximating the implied volatilities and estimating the model parameters are important topics in quantitative finance. This study proposes an approximation formula for short-maturity near-the-money implied volatilities in stochastic volatility models. A general second-order nonlinear PDE for implied volatility is derived in terms of time-to-maturity and log-moneyness from the Feyman-Kac formula. Using regularity conditions and the Taylor expansion, an approximation formula for implied volatility is obtained for short-maturity near-the-money call options in two stochastic volatility models: Heston model and SABR model. In addition, we proposed a novel numerical method to estimate model parameters. This method reduces the number of model parameters that should be estimated. Generating sample data on log-moneyness, time-to-maturity, and implied volatility, we estimate the model parameters fitting the sample data in the above two models. Our method provides parameter estimates that are close to true values.

1. INTRODUCTION

1.1. **Overview.** In this study, we investigate how to approximate implied volatilities and estimate the parameters in the stochastic volatility models. Recently, the number of short-maturity options traded in the market is explosively growing. Evaluating option prices under short-maturity near-the-money circumstance has become crucial, however the implied volatility has no closed-form formula in general. We aim to derive an approximation formula for short-maturity near-the-money implied volatilities in stochastic volatility models: Heston model and

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SABR model. Moreover, we propose a novel numerical method for estimating model parameters. Considering sample data on log-moneyness, time-to-maturity, and implied volatility, we can estimate the model parameters in the above two models using this approximation formula and the least squares method.

We derive an approximation formula for implied volatility in stochastic volatility models as follows. The implied volatility of a European call option is defined as the value of the volatility parameter in the Black–Scholes formula such that the Black–Scholes call price is equal to the actual option price. Let $\sigma = \sigma(\tau, x, \nu)$ be the implied volatility of time-to-maturity τ, logarithmic value x of the strike price divided by the forward stock price and stochastic volatility ν. These notations are further described in later sections. We can obtain an approximation polynomial function $\overline{\sigma}^H$ that satisfies

$$\sigma(\tau, x, \nu) - \overline{\sigma}^H(\tau, x, \nu) = o((|\tau| + x^2)^{3/2} + x^2) \text{ as } \tau, x \to 0$$

for the Heston model and $\overline{\sigma}^S$ that satisfies

$$\sigma(\tau, x, \nu) - \overline{\sigma}^S(\tau, x, \nu) = o(|\tau| + x^2) \text{ as } \tau, x \to 0$$

for the SABR model. Refer to Remark 2.5 for the choice of such approximation orders. To achieve this, we derive a general second-order nonlinear PDE for implied volatility. Using regularity conditions and the Taylor expansion in this general PDE, the approximation formula is derived for short-maturity near-the-money call options in two stochastic volatility models: Heston model and SABR model.

Our proposed method is more improved than previous works in the following ways. Forde et al. [1] derived a small-maturity expansion formula for call option prices and closed-form expansion for implied volatility under the Heston model. Their method can be applied only to the Heston model because their method highly depends on the closed-form solution of the Heston call price. On the other hand, our method can be applied to various models which do not have the closed-form solutions. Floc'h and Kennedy [2] presented a new formula for normal or log-normal volatility, and calibrated the SABR model to particular market volatilities. Their method is valid for data with a single maturity. However, our calibration method can use data with various maturities, and thus we can obtain more accurate model parameters using a larger range of data.

Furthermore, we propose a novel numerical method for estimating the model parameters using the above approximation polynomials $\overline{\sigma}^H$ and $\overline{\sigma}^S$ combined with the least-squares method. We transform the problem of minimizing the mean squared error function into the problem of minimizing elementary functions that can be explicitly expressed in closed-form. For the Heston model, the elementary function is obtained by solving a linear matrix equation, whereas for the SABR model, it is obtained by solving a cubic algebraic equation. This method reduces the number of model parameters that should be estimated. This is a new way of parameter reduction which is not developed before.

The remainder of this paper is organized as follows. In Section 1.2, we provide several related literatures. In Section 2, under the Heston model, we provide an approximation formula for short-maturity at-the-money implied volatilities, and derive a novel numerical method for estimating the model parameters using empirical data. The same analysis is conducted for the SABR model in Section 3. The final section summarizes the study. The technical details are presented in the Appendix.

1.2. **Related literature.** Several authors have investigated methods of approximating implied volatilities. Medvedev and Scaillet [3] derived an asymptotic expansion formula for implied volatility under a two-factor jump-diffusion stochastic volatility model when the time-to-maturity is small. They further proposed a simple calibration procedure for an arbitrary parametric model for short-term near-the-money implied volatilities. In addition, Medvedev [4] derived a formula for the zero time-to-maturity limit of the implied volatilities of European options under stochastic volatility model. Feng et al. [5] studied the Heston stochastic volatility model in a regime where the maturity is small but large compared to the mean reversion time of the stochastic volatility factor. They derived a large deviation principle and computed the rate function through a precise study of the moment-generating function and its asymptotic function. Gatheral et al. [6] calculated the first- and second-order deviations of implied volatility in local volatility models, and obtained approximations, which they numerically demonstrated to be highly accurate.

Many researchers have studied methods for estimating the model parameters. Alós et al. in [7] obtained second-order approximation to the implied volatility for short maturities, and calibrated the full set of parameters of the Heston model accurately by using the approximation. They also provided a quick calibration of a closed-form approximation of vanilla option prices. Cui et al. [8] expressed calibration as a nonlinear least-squares problem, and exploited a suitable representation of the Heston characteristic function.

2. HESTON MODEL

2.1. **PDE derivation.** This section introduces implied volatility in the Heston model [9] and presents a second-order nonlinear PDE for implied volatility. The Heston model on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is expressed as

$$\begin{aligned} d\nu_t &= \kappa(\theta - \nu_t)\, dt + \xi\sqrt{\nu_t}\, dW_t^1\,, \\ dS_t &= rS_t\, dt + \sqrt{\nu_t}S_t\, dW_t^2\,, \\ dW_t^1 dW_t^2 &= \rho\, dt\,, \end{aligned}$$

where $r \geq 0$ is a short rate, $\kappa, \theta, \xi > 0$ and $(W_t^1, W_t^2)_{t\geq 0}$ is a correlated Brownian motion with correlation $-1 \leq \rho \leq 1$ on the filtered probability space. Here, $(S_t)_{t\geq 0}$ is the stock price and $(\nu_t)_{t\geq 0}$ is the square of the volatility process. Let x_t be the log-moneyness of a time-t call price on the stock with strike price K and maturity T, that is, $x_t = \log(K/S_t e^{r(T-t)})$. Then we obtain

$$dx_t = \frac{\nu_t}{2}\, dt - \sqrt{\nu_t}\, dW_t^2\,.$$

The call price is given as

$$\begin{aligned}\text{Call price} &= e^{-r(T-t)}\mathbb{E}[(S_T-K)_+|\mathcal{F}_t]\\ &= Ke^{-r(T-t)}\mathbb{E}[(e^{-x_T}-1)_+|x_t,\nu_t]\\ &= Ke^{-r(T-t)}C(t,x_t,\nu_t)\end{aligned} \tag{2.1}$$

where $C(t,x,\nu) := \mathbb{E}[(e^{-x_T}-1)_+|x_t = x, \nu_t = \nu]$. Meanwhile, the Black–Scholes (BS) formula divided by $Ke^{-r(T-t)}$ is

$$C^{BS}(x,\sigma\sqrt{T-t}) := e^{-x}N\Big(\frac{-x+\sigma^2(T-t)}{2\sigma\sqrt{T-t}}\Big) - N\Big(\frac{-x-\sigma^2(T-t)}{2\sigma\sqrt{T-t}}\Big)\,, \tag{2.2}$$

where x is the log-moneyness of the stock price and N is the cumulative distribution function of the standard normal distribution. By setting

$$C(t,x,\nu) = C^{BS}(x,w)$$

with $\tau = T-t$ and $w = \sigma\sqrt{\tau}$, we can define the implied volatility σ as a function of τ, x, ν.

Proposition 2.1. *In the Heston model, we let $\sigma = \sigma(\tau,x,\nu)$ be the implied volatility function. The function σ is infinitely differentiable and satisfies*

$$\begin{aligned}0 =& -\sigma^3\sigma_\tau\tau - \frac{1}{2}\sigma^4 + \frac{1}{2}\nu\big(\sigma^2 - 2x\sigma\sigma_x + x^2\sigma_x^2 - \frac{1}{4}\sigma^4\sigma_x^2\tau^2 + \sigma^3\sigma_{xx}\tau\big)\\ &+ \kappa(\theta-\nu)\sigma^3\sigma_\nu\tau + \frac{1}{2}\xi^2\nu\big(x^2\sigma_\nu^2 - \frac{1}{4}\sigma^4\sigma_\nu^2\tau^2 + \sigma^3\sigma_{\nu\nu}\tau\big)\\ &- \rho\xi\nu\big(-x\sigma\sigma_\nu - \frac{1}{2}\sigma^3\sigma_\nu\tau + x^2\sigma_\nu\sigma_x - \frac{1}{4}\sigma^4\sigma_\nu\sigma_x\tau^2 + \sigma^3\sigma_{x\nu}\tau\big)\,.\end{aligned} \tag{2.3}$$

Proof. First, we ensure that σ is infinitely differentiable. In our formulation, σ is a solution to

$$F(t,x,\nu,\sigma) = C(t,x,\nu) - C^{BS}(x,\sigma\sqrt{T-t}) = 0\,.$$

We can show the two functions C^{BS} and C are infinitely differentiable using (2.2) and Appendix A, respectively. According to the implicit function theorem, the function σ is infinitely differentiable in t, x, ν (thus, in τ, x, ν).

Next, we derive a PDE for implied volatility σ. Observe that

$$\begin{aligned}dC(t,x_t,\nu_t) =& \Big(C_t + \frac{1}{2}\nu_t(C_x + C_{xx}) + \kappa(\theta-\nu_t)C_\nu + \frac{1}{2}\xi^2\nu_tC_{\nu\nu} - \rho\xi\nu_tC_{x\nu}\Big)\,dt\\ &+ \sqrt{\nu_t}C_x\,dW_t^2 + \xi\sqrt{\nu_t}C_\nu\,dW_t^1\,,\end{aligned}$$

which is obtained by applying Itô's formula. The martingale property of the value process $C(t,x_t,\nu_t) = \mathbb{E}[(e^{-x_T}-1)_+|\mathcal{F}_t]$, $0 \le t \le T$ requires the drift term to be zero, which induces

$$\begin{aligned}&C_t + \frac{1}{2}\nu(C_x + C_{xx}) + \kappa(\theta-\nu)C_\nu + \frac{1}{2}\xi^2\nu C_{\nu\nu} - \rho\xi\nu C_{x\nu} = 0\,,\\ &C(T,x_T,\,\cdot\,) = (e^{-x_T}-1)_+\,.\end{aligned} \tag{2.4}$$

The partial derivatives of C can be written as

$$C_t = -C_w^{BS} w_\tau,\ C_x = C_x^{BS} + C_w^{BS} w_x,\ C_{xx} = C_{xx}^{BS} + 2C_{xw}^{BS} w_x + C_{ww}^{BS} w_x^2 + C_w^{BS} w_{xx}\,,$$
$$C_\nu = C_w^{BS} w_\nu,\ C_{\nu\nu} = C_{ww}^{BS} w_\nu^2 + C_w^{BS} w_{\nu\nu},\ C_{x\nu} = C_{xw}^{BS} w_\nu + C_w^{BS} w_{x\nu} + C_{ww}^{BS} w_x w_\nu\,.$$

Meanwhile, the partial derivatives of C^{BS} are

$$C_x^{BS} = -e^{-x} N(d_1)\,,\ C_w^{BS} = e^{-x} N'(d_1)\,,\ C_{xx}^{BS} = e^{-x} N(d_1) + C_w^{BS} \frac{1}{w}\,,$$
$$C_{xw}^{BS} = C_w^{BS}\Big(-\frac{x}{w^2} - \frac{1}{2}\Big)\,,\ C_{ww}^{BS} = C_w^{BS}\Big(\frac{x^2}{w^3} - \frac{w}{4}\Big)\,,$$

where $d_1 = \frac{-x+\sigma^2(T-t)}{2\sigma\sqrt{T-t}}$. Using these equalities, we obtain

$$\begin{aligned}
&C_t = -C_w^{BS} w_\tau,\\
&C_x + C_{xx} = C_w^{BS}\Big(\frac{1}{w} - \frac{2xw_x}{w^2} + (\frac{x^2}{w^3} - \frac{w}{4})w_x^2 + w_{xx}\Big)\,,\\
&C_\nu = C_w^{BS} w_\nu,\\
&C_{\nu\nu} = C_w^{BS}\Big((\frac{x^2}{w^3} - \frac{w}{4})w_\nu^2 + w_{\nu\nu}\Big)\,,\\
&C_{x\nu} = C_w^{BS}\Big((-\frac{x}{w^2} - \frac{1}{2})w_\nu + w_{x\nu} + (\frac{x^2}{w^3} - \frac{w}{4})w_x w_\nu\Big)\,.
\end{aligned}$$

By substituting these in the PDE and canceling out the common term C_w^{BS}, we obtain the PDE of w

$$\begin{aligned}
&- w_\tau + \frac{1}{2}\nu\Big(\frac{1}{w} - \frac{2xw_x}{w^2} + (\frac{x^2}{w^3} - \frac{w}{4})w_x^2 + w_{xx}\Big) + \kappa(\theta-\nu)w_\nu\\
&+ \frac{1}{2}\xi^2\nu\Big((\frac{x^2}{w^3} - \frac{w}{4})w_\nu^2 + w_{\nu\nu}\Big) - \rho\xi\nu\Big((-\frac{x}{w^2} - \frac{1}{2})w_\nu + w_{x\nu} + (\frac{x^2}{w^3} - \frac{w}{4})w_x w_\nu\Big) = 0\,.
\end{aligned}$$

Finally, by substituting $\sigma(\tau,x,\nu)\sqrt{\tau}$ for w and multiplying both sides by $\sigma^3\sqrt{\tau}$, we obtain the desired PDE. □

2.2. **Approximation formula.** We provide an approximation formula for short-maturity near-the-money implied volatilities in the Heston model.

Proposition 2.2. *In the Heston model, we let $\sigma = \sigma(\tau,x,\nu)$ be the implied volatility function. The polynomial function*

$$\begin{aligned}
\overline{\sigma}^H(\tau,x,\nu) =&\sqrt{\nu} + \frac{\rho\xi}{4\sqrt{\nu}}x + \Big(\frac{1}{8}\rho\xi\sqrt{\nu} + \frac{1}{4\sqrt{\nu}}\kappa(\theta-\nu) + \frac{1}{96\sqrt{\nu}}(\rho^2-4)\xi^2\Big)\tau\\
&+ \frac{(2-5\rho^2)\xi^2}{48\nu\sqrt{\nu}}x^2 + \Big(-\frac{\rho^2\xi^2}{96\sqrt{\nu}} + \frac{\kappa\rho\xi}{48\sqrt{\nu}} - \frac{5\kappa\theta\rho\xi}{48\nu\sqrt{\nu}} + \frac{(4-3\rho^2)\rho\xi^3}{128\nu\sqrt{\nu}}\Big)x\tau
\end{aligned}$$

satisfies

$$\sigma(\tau,x,\nu) - \overline{\sigma}^H(\tau,x,\nu) = o((|\tau| + x^2)^{3/2} + x^2) \text{ as } \tau, x \to 0\,.$$

Proof. From Proposition 2.1, we know that the function σ is infinitely differentiable, and thus, we can apply the Taylor expansion. A polynomial function $\overline{\sigma}^H$ that satisfies

$$\sigma(\tau, x, \nu) - \overline{\sigma}^H(\tau, x, \nu) = o((|\tau| + x^2)^{3/2} + x^2) \text{ as } \tau, x \to 0$$

is expressed as

$$\begin{aligned} &\overline{\sigma}^H(\tau, x, \nu) \\ &= \overline{\sigma}^H(0,0,\nu) + \frac{\partial \overline{\sigma}^H}{\partial x}(0,0,\nu)x + \frac{\partial \overline{\sigma}^H}{\partial \tau}(0,0,\nu)\tau + \frac{1}{2}\frac{\partial^2 \overline{\sigma}^H}{\partial x^2}(0,0,\nu)x^2 + \frac{\partial^2 \overline{\sigma}^H}{\partial x \partial \tau}(0,0,\nu)x\tau\,. \end{aligned}$$

The partial derivatives in this equation are computed as follows. Letting $x, \tau \to 0$ in the PDE (2.3),

$$0 = -\frac{1}{2}\sigma^4 + \frac{1}{2}\nu\sigma^2\,,$$

thus, $\sigma(0,0,\nu) = \sqrt{\nu}$. Differentiating both sides of the PDE in x, we obtain

$$0 = -2\sigma^3\sigma_x + \rho\xi\nu\sigma\sigma_\nu,$$

which gives $\sigma_x(0,0,\nu) = \frac{\rho\xi}{4\sqrt{\nu}}$. Similarly, differentiating the PDE in x twice and thrice,

$$\sigma_{xx}(0,0,\nu) = \frac{(2-5\rho^2)\xi^2}{24\nu\sqrt{\nu}}\,,\ \sigma_{xxx}(0,0,\nu) = \frac{(8\rho^2-5)\rho\xi^3}{16\nu^2\sqrt{\nu}}\,.$$

Next, we conduct the same procedure for variable τ. Differentiating the PDE in τ and letting $x, \tau \to 0$, we have that

$$\sigma_\tau(0,0,\nu) = \frac{1}{8}\rho\xi\sqrt{\nu} + \frac{1}{4\sqrt{\nu}}\kappa(\theta-\nu) + \frac{1}{96\sqrt{\nu}}(\rho^2-4)\xi^2\,.$$

Differentiating the PDE in x, τ and letting $x, \tau \to 0$, we obtain

$$\sigma_{x\tau}(0,0,\nu) = -\frac{\rho^2\xi^2}{96\sqrt{\nu}} + \frac{\kappa\rho\xi}{48\sqrt{\nu}} - \frac{5\kappa\theta\rho\xi}{48\nu\sqrt{\nu}} + \frac{(4-3\rho^2)\rho\xi^3}{128\nu\sqrt{\nu}}\,.$$

This yields the desired result.

□

Remark 2.3. *A similar approximation polynomial for the squared implied volatility function* $(\overline{\sigma}^H)^2$ *was computed in* [1]. *They obtained this result by analyzing asymptotic behavior for Heston call options. On the other hand, we derived an approximation polynomial for the implied volatility function* $\overline{\sigma}^H$ *itself by analyzing the general second-order nonlinear PDE in Proposition 2.1.*

Remark 2.4. *In the above proposition, the approximation formula is of the order* $o((|\tau| + x^2)^{3/2} + x^2)$, *however, we can attain a higher-order approximation by computing higher-order derivatives recursively using the same method.*

2.3. **Parameter estimation.** Using the volatility approximation result, we develop a novel estimator for the model parameters from short-maturity near-the-money call prices. We formulate a subproblem as a linear equation, instead of optimizing all parameters simultaneously, and then determine the best parameters $(\nu, \kappa, \theta, \rho, \xi)$ that fit the particular short-maturity near-the-money market data $(x_i, \sigma_i, \tau_i)_{i=1,2,\dots,N}$. We aim to minimize the mean squared error (MSE) function

$$\Phi^H(\nu, \kappa, \theta, \rho, \xi) = \sum_{i=1}^{N} w_i^2 \left\| \sigma_i - \overline{\sigma}^H(x_i, \tau_i, \nu\,; \kappa, \theta, \rho, \xi) \right\|^2 .$$

for weights $w_1, w_2, \cdots, w_N$ over $(\nu, \kappa, \theta, \rho, \xi) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times [-1, 1] \times \mathbb{R}^+$. For simplicity, we define variables $\gamma = \kappa\theta$, $\eta = \rho\xi$ and $\phi = \xi^2$. Then the approximation function $\overline{\sigma}^H(x_i, \tau_i, \nu) = \overline{\sigma}^H(x_i, \tau_i, \nu\,; \kappa, \theta, \rho, \xi)$ can be written as

$$\begin{aligned}
\overline{\sigma}^H(x_i, \tau_i, \nu) &= \sqrt{\nu} + \frac{\rho\xi}{4\sqrt{\nu}} x_i + \Big(\frac{1}{8}\rho\xi\sqrt{\nu} + \frac{1}{4\sqrt{\nu}}\kappa(\theta - \nu) + \frac{1}{96\sqrt{\nu}}(\rho^2 - 4)\xi^2 \Big)\tau_i \\
&\quad + \frac{(2 - 5\rho^2)\xi^2}{48\nu\sqrt{\nu}} x_i^2 + \Big(- \frac{\rho^2\xi^2}{96\sqrt{\nu}} + \frac{\kappa\rho\xi}{48\sqrt{\nu}} - \frac{5\kappa\theta\rho\xi}{48\nu\sqrt{\nu}} + \frac{(4 - 3\rho^2)\rho\xi^3}{128\nu\sqrt{\nu}} \Big) x_i\tau_i \\
&= \sqrt{\nu} + \frac{x_i}{4\sqrt{\nu}}\eta + \frac{\tau_i\sqrt{\nu}}{8}\eta + \frac{\tau_i}{96\sqrt{\nu}}\eta^2 - \frac{5x_i^2}{48\nu\sqrt{\nu}}\eta^2 - \frac{x_i\tau_i}{96\sqrt{\nu}}\eta^2 - \frac{3x_i\tau_i}{128\nu\sqrt{\nu}}\eta^3 \\
&\quad + \Big(\frac{\tau_i}{4\sqrt{\nu}} - \frac{5x_i\tau_i\eta}{48\nu\sqrt{\nu}} \Big)\gamma + \Big(- \frac{\tau_i\sqrt{\nu}}{4} + \frac{x_i\tau_i\eta}{48\sqrt{\nu}} \Big)\kappa + \Big(\frac{4x_i^2 - 4\tau_i + 3x_i\tau_i\eta}{96\nu\sqrt{\nu}} \Big)\phi \\
&= a_i + b_i\gamma + c_i\kappa + d_i\phi\,.
\end{aligned}$$

The meaning of a_i, b_i, c_i, d_i is straightforward. Define $A_i = a_i w_i$, $B_i = b_i w_i$, $C_i = c_i w_i$, $D_i = d_i w_i$ and

$$\Psi^H(\nu, \eta, \gamma, \kappa, \phi) := \sum_{i=1}^{N} w_i^2 \left\| \sigma_i - \overline{\sigma}^H(x_i, \tau_i, \nu) \right\|^2 = \sum_{i=1}^{N} \left\| \sigma_i w_i - (A_i + B_i\gamma + C_i\kappa + D_i\phi) \right\|^2 .$$

The problem of minimizing the MSE function can be expressed as

$$\begin{aligned}
\inf_{\nu,\eta,\gamma,\kappa,\phi} \Psi^H(\nu, \eta, \gamma, \kappa, \phi) &= \inf_{\nu,\eta} \inf_{\gamma,\kappa,\phi} \Psi^H(\nu, \eta, \gamma, \kappa, \phi) \\
&= \inf_{\nu,\eta} \inf_{\gamma,\kappa,\phi} \sum_{i=1}^{N} \left\| \sigma_i w_i - (A_i + B_i\gamma + C_i\kappa + D_i\phi) \right\|^2 .
\end{aligned}$$

For each ν and η, we can apply the linear regression method to

$$\inf_{\gamma,\kappa,\phi} \sum_{i=1}^{N} \left\| \sigma_i w_i - (A_i + B_i\gamma + C_i\kappa + D_i\phi) \right\|^2 .$$

The first-order condition with respect to (γ, κ, ϕ) leads to the least-squares solution

$$\begin{pmatrix} \gamma^{\nu,\eta} \\ \kappa^{\nu,\eta} \\ \phi^{\nu,\eta} \end{pmatrix} = \begin{pmatrix} \sum {B_i}^2 & \sum B_i C_i & \sum B_i D_i \\ \sum B_i C_i & \sum C_i^2 & \sum C_i D_i \\ \sum B_i D_i & \sum C_i D_i & \sum {D_i}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum B_i(\sigma_i w_i - A_i) \\ \sum C_i(\sigma_i w_i - A_i) \\ \sum D_i(\sigma_i w_i - A_i) \end{pmatrix} \tag{2.5}$$

assuming that 3×3 square matrix on the right-hand side is non-singular. Defining $F^H(\nu, \eta) := \Psi^H(\nu, \eta, \gamma^{\nu,\eta}, \kappa^{\nu,\eta}, \phi^{\nu,\eta})$, the minimization problem becomes

$$\inf_{\nu,\eta,\gamma,\kappa,\phi} \Psi^H(\nu, \eta, \gamma, \kappa, \phi) = \inf_{\nu,\eta} \inf_{\gamma,\kappa,\phi} \Psi^H(\nu, \eta, \gamma, \kappa, \phi) = \inf_{\nu,\eta} F^H(\nu, \eta)$$

over $(\nu, \eta) \in \mathbb{R}^+ \times \mathbb{R}$. Consequently, the minimization problem over the five variables is reduced to the minimization problem over the two variables.

Remark 2.5. *The $((|\tau| + x^2)^{3/2} + x^2)$-order approximation formula is the lowest degree polynomial for which our estimation method is valid. The least-squares solution obtained from the $(|\tau| + x^2)$-order approximation polynomial is not uniquely determined because 3×3 matrix in (2.5) is always singular. In addition, the linear regression method cannot be applied due to the quadratic terms of parameters in the τ^2-coefficient of the $(\tau^2 + x^2)$-order approximation polynomial.*

2.4. **Calibration.** In this section, we calibrate the model parameters using the sample data. For the Heston model, we assume that the short rate is zero, and that $\nu_0 = 0.04$, $\kappa = 3$, $\theta = 0.04$, $\xi = 0.2$, $\rho = -0.2$. Using the Andersen-Lake pricing method in [11], $N = 45$ call option data with time-to-maturity $\tau \in \mathcal{T} = \{0.02, 0.04, 0.06, 0.08, 0.10\}$ and log-moneyness $x \in \mathcal{X} = \{0, \pm 0.01, \pm 0.02, \pm 0.03, \pm 0.04\}$ are generated. Let

$$\{(\tau_i, x_i) : i = 1, 2, \cdots, N\} = \mathcal{T} \times \mathcal{X}$$

and σ_i be the implied volatility corresponding to (τ_i, x_i) for $i = 1, 2, \cdots, N$. The implied volatility surface is presented in Fig. 1. Using the approximation method with weights $w_i = 1/(\sigma_i e^{15(\tau_i + |x_i|)})$, the estimated parameters are $\nu_0 = 0.040$, $\kappa = 2.995$, $\theta = 0.048$, $\xi = 0.193$, $\rho = -0.216$, which are close to the true values.

3. SABR MODEL

3.1. **PDE derivation.** We conduct similar steps to the SABR model, and derive a PDE for volatility in this model. The SABR model on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is expressed by

$$\begin{aligned} d\nu_t &= \alpha \nu_t \, dW_t^1 \, , \\ dF_t &= \nu_t F_t^\beta \, dW_t^2 \, , \\ dW_t^1 dW_t^2 &= \rho dt \, , \end{aligned}$$

where $\alpha > 0$, $0 \leq \beta \leq 1$ and $(W_t^1, W_t^2)_{t \geq 0}$ is a correlated Brownian motion with correlation $-1 \leq \rho \leq 1$ on the filtered probability space. Here, $(F_t)_{t \geq 0}$ is the forward stock price and

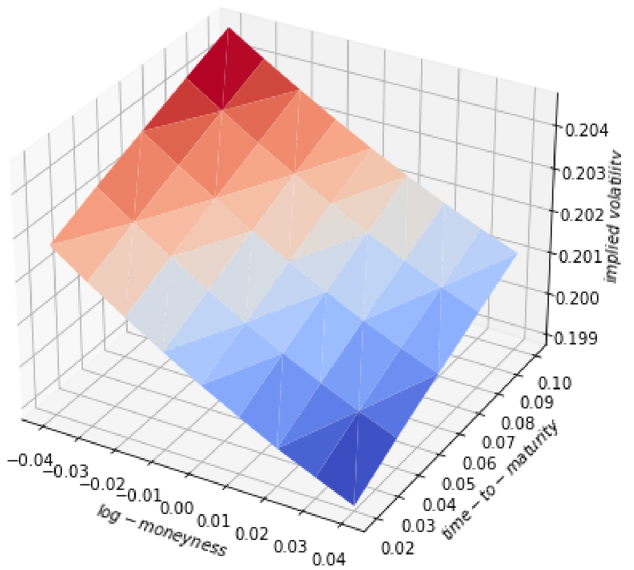

FIGURE 1. The implied volatility surface for the Heston model

$(\nu_t)_{t\geq 0}$ is the volatility process. Let x_t be the log-moneyness of a time-t call price on the stock with strike price K and maturity T, that is, $x_t = \log(K/F_t)$. Then we have

$$dx_t = \frac{1}{2}\nu_t^2 K^{2\beta-2} e^{(2-2\beta)x_t}\, dt - \nu_t K^{\beta-1} e^{(1-\beta)x_t}\, dW_t^2 .$$

Applying the same procedure used in 2.1, we obtain the following proposition.

Proposition 3.1. *In the SABR model, let $\sigma = \sigma(\tau, x, \nu)$ be the implied volatility function. Then the function σ is infinitely differentiable and satisfies*

$$\begin{aligned} 0 =& -\sigma^3\sigma_\tau\tau - \frac{1}{2}\sigma^4 + \frac{1}{2}\nu^2 K^{2\beta-2} e^{(2-2\beta)x}\big(\sigma^2 - 2x\sigma\sigma_x + x^2\sigma_x^2 - \frac{1}{4}\sigma^4\sigma_x^2\tau^2 + \sigma^3\sigma_{xx}\tau\big) \\ &+ \frac{1}{2}\alpha^2\nu^2\big(x^2\sigma_\nu^2 - \frac{1}{4}\sigma^4\sigma_\nu^2\tau^2 + \sigma^3\sigma_{\nu\nu}\tau\big) \\ &- \rho\alpha\nu^2 K^{\beta-1} e^{(1-\beta)x}\Big(- x\sigma\sigma_\nu - \frac{1}{2}\sigma^3\sigma_\nu\tau + x^2\sigma_\nu\sigma_x - \frac{1}{4}\sigma^4\sigma_\nu\sigma_x\tau^2 + \sigma^3\sigma_{x\nu}\tau\Big) . \end{aligned}$$

3.2. **Approximation formula.** An approximation formula for short-maturity near-the-money implied volatilities in the SABR model is derived. The proof of the following proposition is similar to that in Proposition 2.2.

Proposition 3.2. *In the SABR model, we let $\sigma = \sigma(\tau, x, \nu)$ be the implied volatility function. Then the polynomial function*

$$\begin{aligned}\overline{\sigma}^S(\tau, x, \nu) &= \nu K^{\beta-1} \\ &\quad + \Big(\frac{1-\beta}{2}\nu K^{\beta-1} + \frac{1}{2}\alpha\rho\Big)x + \Big(\frac{(1-\beta)^2}{12}\nu K^{\beta-1} + \frac{2-3\rho^2}{12\nu}\alpha^2 K^{1-\beta}\Big)x^2 \\ &\quad + \Big(\nu^2 K^{2\beta-2}\Big(\frac{(1-\beta)^2}{24}\nu K^{\beta-1} + \frac{2-3\rho^2}{24\nu}\alpha^2 K^{1-\beta}\Big) + \frac{\beta}{4}\rho\alpha\nu^2 K^{2\beta-2}\Big)\tau\end{aligned}$$

satisfies

$$\sigma(\tau, x, \nu) - \overline{\sigma}^H(\tau, x, \nu) = o(|\tau| + x^2) \text{ as } \tau, x \to 0\,.$$

Remark 3.3. *The first-order approximation for short-maturity at-the-money (i.e. $\tau \to 0$, $x = 0$) implied volatilities was computed in* [10]. *This proposition generalizes their result to short-maturity near-the-money (i.e. $\tau \to 0$, $x \to 0$) implied volatilities.*

3.3. **Parameter estimation.** Using the volatility approximation result, we develop a novel estimator for the model parameters from short-maturity near-the-money call prices. We formulate a subproblem as a cubic equation, instead of optimizing all parameters simultaneously. The main purpose is to determine the best parameters $(\nu, \alpha, \beta, \rho)$ that fit the particular short-maturity near-the-money sample data $(K_i, x_i, \sigma_i, \tau_i)_{i=1,2,\dots,N}$. We aim to minimize the mean squared error (MSE) function

$$\Phi^S(\nu, \beta, \alpha, \rho) = \sum_{i=1}^{N} w_i^2 \big\|\sigma_i - \overline{\sigma}^S(\tau_i, x_i, \nu\,; \beta, \alpha, \rho)\big\|^2$$

for weights $w_1, w_2, \cdots, w_N$ over $(\nu, \beta, \alpha, \rho) \in \mathbb{R}^+ \times [0,1] \times \mathbb{R}^+ \times [-1,1]$. Define $\mu = \alpha^2$, and $\lambda = \alpha\rho$, then

$$\begin{aligned}&\overline{\sigma}^S(\tau_i, x_i, \nu) \\ &= \nu K_i^{\beta-1} + \Big(\frac{1-\beta}{2}\nu K_i^{\beta-1} + \frac{1}{2}\alpha\rho\Big)x_i + \Big(\frac{(1-\beta)^2}{12}\nu K_i^{\beta-1} + \frac{2-3\rho^2}{12\nu}\alpha^2 K_i^{1-\beta}\Big)x_i^2 \\ &\quad + \Big(\nu^2 K_i^{2\beta-2}\Big(\frac{(1-\beta)^2}{24}\nu K_i^{\beta-1} + \frac{2-3\rho^2}{24\nu}\alpha^2 K_i^{1-\beta}\Big) + \frac{\beta}{4}\rho\alpha\nu^2 K_i^{2\beta-2}\Big)\tau_i \\ &= \nu K_i^{\beta-1} + \frac{(1-\beta)^2\nu^3 K_i^{3\beta-3}}{24}\tau_i + \frac{1-\beta}{2}x_i\nu K_i^{\beta-1} + \frac{(1-\beta)^2 x_i^2}{12}\nu K_i^{\beta-1} \\ &\quad + \Big(\frac{x_i^2}{6\nu}K_i^{1-\beta} + \frac{\nu\tau_i K_i^{\beta-1}}{12}\Big)\mu + \Big(\frac{1}{2}x_i + \frac{\beta}{4}\nu^2\tau_i K_i^{2\beta-2}\Big)\lambda - \Big(\frac{\nu\tau_i K_i^{\beta-1}}{8} + \frac{x_i^2}{4\nu}K_i^{1-\beta}\Big)\lambda^2 \\ &= a_i + b_i\mu + c_i\lambda + d_i\lambda^2\,.\end{aligned}$$

The meaning of a_i, b_i, c_i, d_i is straightforward. Define $A_i = a_i w_i$, $B_i = b_i w_i$, $C_i = c_i w_i$, $D_i = d_i w_i$ and

$$\Psi^S(\nu,\beta,\mu,\lambda) := \sum_{i=1}^N w_i^2 \left\| \sigma_i - \overline{\sigma}^S(\tau_i, x_i, \nu) \right\|^2 = \sum_{i=1}^N \left\| \sigma_i w_i - (A_i + B_i\mu + C_i\lambda + D_i\lambda^2) \right\|^2 .$$

The problem of minimizing the MSE function can be expressed as

$$\begin{aligned} \inf_{\nu,\beta,\mu,\lambda} \Psi^S(\nu,\beta,\mu,\lambda) &= \inf_{\nu,\beta} \inf_{\mu,\lambda} \Psi^S(\nu,\beta,\mu,\lambda) \\ &= \inf_{\nu,\beta} \inf_{\mu,\lambda} \sum_{i=1}^N \left\| \sigma_i w_i - (A_i + B_i\mu + C_i\lambda + D_i\lambda^2) \right\|^2 . \end{aligned}$$

For each ν and β, from the first-order condition on μ,

$$\frac{\partial}{\partial\mu}\Psi^S(\nu,\beta,\mu,\lambda) = -2\sum_{i=1}^N B_i\big(\sigma_i w_i - A_i - B_i\mu - C_i\lambda - D_i\lambda^2\big) = 0\,,$$

we have

$$\mu = P + Q\lambda + R\lambda^2 \tag{3.1}$$

where

$$P = \frac{\sum_{i=1}^N B_i(\sigma_i w_i - A_i)}{\sum_{i=1}^N B_i^2}\,,\ Q = -\frac{\sum_{i=1}^N B_i C_i}{\sum_{i=1}^N B_i^2}\,,\ R = -\frac{\sum_{i=1}^N B_i D_i}{\sum_{i=1}^N B_i^2}\,.$$

Moreover, from the first-order condition on λ,

$$\frac{\partial}{\partial\lambda}\Psi^S(\nu,\beta,\mu,\lambda) = -\sum_{i=1}^N \big(C_i + 2D_i\lambda\big)\big(\sigma_i w_i - A_i - B_i\mu - C_i\lambda - D_i\lambda^2\big) = 0\,,$$

and by combining this with (3.1), we obtain the cubic equation for λ

$$\begin{aligned} &\Big(-2R\sum_{i=1}^N B_iD_i - 2\sum_{i=1}^N D_i^2\Big)\lambda^3 + \Big(-3\sum_{i=1}^N C_iD_i - R\sum_{i=1}^N B_iC_i - 2Q\sum_{i=1}^N B_iD_i\Big)\lambda^2 \\ &+ \Big(-\sum_{i=1}^N C_i^2 - Q\sum_{i=1}^N B_iC_i + 2\sum_{i=1}^N D_i(\sigma_i w_i - A_i) - 2P\sum_{i=1}^N B_iD_i\Big)\lambda \\ &+ \Big(\sum_{i=1}^N C_i(\sigma_i w_i - A_i) - P\sum_{i=1}^N B_iC_i\Big) = 0\,. \end{aligned}$$

For each ν and β, because the above equation has at most three real solutions, the minimization problem can be easily solved by evaluating the MSE for the three solutions. We denote the minimizer by $(\mu^{\nu,\beta}, \lambda^{\nu,\beta})$ and define $F^S(\nu,\beta) := \Psi^S(\nu,\beta,\mu^{\nu,\beta},\lambda^{\nu,\beta})$. The minimization problem becomes

$$\inf_{\nu,\beta,\mu,\lambda} \Psi^S(\nu,\beta,\mu,\lambda) = \inf_{\nu,\beta}\inf_{\mu,\lambda} \Psi^S(\nu,\beta,\mu,\lambda) = \inf_{\nu,\beta} F^S(\nu,\beta)\,.$$

Consequently, the minimization problem over the four variables is reduced to the minimization problem over the two variables.

3.4. **Parameter estimation.** In this section, we calibrate the model parameters using the sample data. For the SABR model, we assume that the short rate is zero, and that $\nu_0 = 0.8$, $\alpha = 0.2$, $\beta = 0.7$, $\rho = -0.4$. The implied volatilities of $N = 100$ call option data with time-to-maturity

$$\tau \in \mathcal{T} = \{0.02, 0.04, 0.06, 0.08, 0.10\}$$

and log-moneyness

$$x \in \mathcal{X} = \{-0.5, -0.45, \cdots, 0.40, 0.45\}$$

can be obtained numerically by Python package `QuantLib`, which is an open source library used for quantative finance. Let

$$\{(\tau_i, x_i) : i = 1, 2, \cdots, N\} = \mathcal{T} \times \mathcal{X}\,.$$

The implied volatility surface is presented in Fig. 2. Using the approximation method with the uniform weights, the estimated parameters are $\nu_0 = 0.84$, $\alpha = 0.195$ $\beta = 0.690$, $\rho = -0.385$, which are close to the true values.

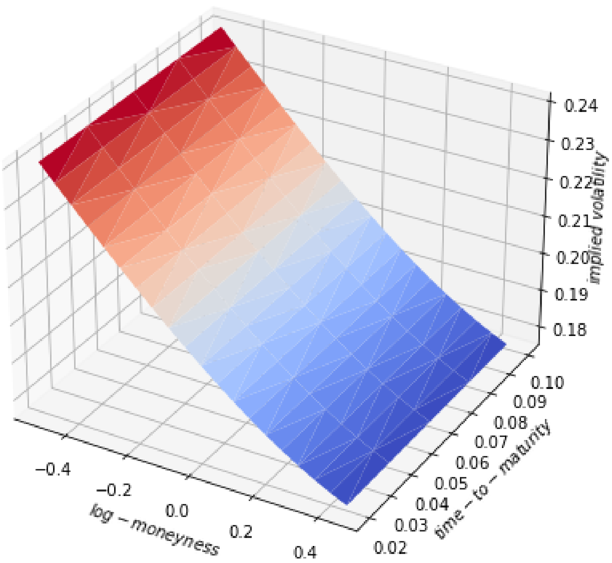

FIGURE 2. The implied volatility surface for the SABR model

4. Conclusion

In this study, we derived an approximation formula for implied volatilities, and then employed it to estimate the model parameters. For short-maturity near-the-money implied volatilities, approximation formulas of order $(|\tau| + x^2)^{3/2} + x^2$ and $|\tau| + x^2$ as $\tau, x \to 0$ were computed for the Heston and SABR models, respectively. To achieve this, a general second-order nonlinear PDE for the implied volatility was derived using the Feyman-Kac formula. By employing regularity conditions and the Taylor expansion, we obtained the approximation formula for the implied volatility.

We provided a novel numerical method for estimating the model parameters. This method reduces the number of model parameters that should be estimated. The least-squares problem over five variables is reduced to the minimization problem over two variables in the Heston model. Similarly, the least squares problem over four variables is reduced to the minimization problem over two variables in the SABR model. Generating sample data on log-moneyness, time-to-maturity, and implied volatility, we estimated the model parameters fitting the sample data in the above two models. Our method provides parameter estimates that are close to true values.

Appendix A. Proof of the regularity of the implied volatility

In this section, we investigate the regularity of solutions to parabolic PDEs, and show that the call price function C in (2.1) is infinitely differentiable. Let D be a bounded domain in $\mathbb{R}^{d+1}$ consisting of a time variable and d-dimensional state variables. For a_{ij}, b_i, c, f, u : $D \to \mathbb{R}$, consider a parabolic operator L defined as

$$Lu = \sum_{i,j}^{n} a_{ij}(t,x)\frac{\partial u}{\partial x_i x_j} + \sum_{i}^{n} b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u - \frac{\partial u}{\partial t} = f\,.$$

Theorem A.1. *Assume that*

$$D_x^m a_{ij}, \quad D_x^m b_i, \quad D_x^m c, \quad D_x^m f \qquad (0 \le m \le p)$$

are Hölder continuous (exponent α) in D. If u is a solution to $Lu = f$ in D, then

$$D_x^m u, \quad D_t D_x^m u \qquad (0 \le m \le p+2, 0 \le k \le p)$$

exist and are Hölder continuous (exponent α) in D.

Theorem A.2. *Assume that*

$$D_t^k D_x^m a_{ij}, \quad D_t^k D_x^m b_i, \quad D_t^k D_x^m c, \quad D_t^k D_x^m f \qquad (0 \le m+2k \le p, k \le q)$$

are Hölder continuous (exponent α) in D. If u is a solution to $Lu = f$ in D, then

$$D_t^k D_x^m u, \qquad (0 \le m+2k \le p+2, k \le q+1)$$

exist and are Hölder continuous (exponent α) in D.

The proof of the two theorems above are on [12, pp.72-74].

Using these theorems, we can prove that the function C in (2.1) is infinitely differentiable. Observe that C is a solution to parabolic PDE (2.4), in which the function a_{ij} satisfies the ellipticity condition and $c \equiv f \equiv 0$. Considering the domain $D_n = (0, T) \times \mathbb{R} \times (1/n, n)$ for $n \in \mathbb{N}$, we can easily verify that the coefficients satisfy all conditions in the theorems. Hence, the solution on the domain D_n is infinitely differentiable. Because differentiability is a local property, we obtain the desired result.

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