Strong limit theorems for partial sums of a random sequence

Yong-Kab Choi\textsuperscript{1}, Tae-Sung Kim\textsuperscript{2} and Soo Hak Sung\textsuperscript{3}

\textsuperscript{1}Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, KOREA, E-mail: mathykcc@gnu.ac.kr
\textsuperscript{2}Department of Mathematics, Wonkwang University, Iksan 570-749, KOREA E-mail: starkim@wonkwang.ac.kr
\textsuperscript{3}Department of Applied Mathematics, Pai Chai University, Taejon 302-735, KOREA E-mail: sungsh@pcu.ac.kr

Abstract

Let \( \{\xi_j; j \geq 1\} \) be a centered strictly stationary random sequence defined by \( S_0 = 0, S_n = \sum_{j=1}^{n} \xi_j \) and \( \sigma(n) = \sqrt{ES^2_n} \), where \( \sigma(t), t > 0, \) is a nondecreasing continuous regularly varying function. Suppose that there exists \( n_0 \geq 1 \) such that, for any \( n \geq n_0 \) and \( 0 \leq \varepsilon < 1 \), there exist positive constants \( c_1 \) and \( c_2 \) such that
\[
\frac{c_1}{\sigma(n)} e^{-(1+\varepsilon)x^2/2} \leq P\left\{ \frac{|S_n|}{\sigma(n)} \geq x \right\} \leq \frac{c_2}{\sigma(n)} e^{-(1-\varepsilon)x^2/2}, \quad x \geq 1.
\]
Under some additional conditions, we investigate strong limit theorems for increments of partial sum processes of the sequence \( \{\xi_j; j \geq 1\} \).

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1. Introduction

Let \( \{X, X_n, n \geq 1\} \) be a sequence of nondegenerate centered independent and identically distributed (i.i.d.) random variables on an underlying probability space.

\textsuperscript{1}Corresponding author.
\( (\Omega, \mathfrak{F}, P) \) such that \( EX^2I\{|X| \leq x\} \) is slowly varying as \( x \to \infty \). Put

\[
S_n = \sum_{i=1}^{n} X_i, \quad V_n^2 = \sum_{i=1}^{n} X_i^2, \quad n \geq 1.
\]

Shao [18] proved the following: For arbitrary \( 0 < \varepsilon < 1/2 \), there exist \( 0 < \delta < 1 \), \( x_0 > 1 \) and \( n_0 \) such that, for any \( n \geq n_0 \) and \( x_0 < x < \delta \sqrt{n} \),

\[
e^{-\left(1+\varepsilon\right)x^2/2} \leq P\left\{ \frac{S_n}{V_n} \geq x \right\} \leq e^{-\left(1-\varepsilon\right)x^2/2}
\]

in Remark 4.1 of the just mentioned paper [18]. Further, Csörgő et al. [4] established a weak invariance principle related to the inequality (1.1) for self-normalized partial sum processes under the assumption that \( X \) belongs to the domain of attraction of the normal law.

On the other hand, consider a sequence of dependent random variables \( \{Y_n; n \geq 1\} \). The sequence \( \{Y_n; n \geq 1\} \) is said to be \textit{positively associated} (PA) if, for any finite subsets \( A, B \) of \( \{1, 2, \cdots\} \) and coordinatewise increasing functions \( f \) and \( g \), we have \( \text{Cov}(f(Y_i; i \in A), g(Y_j; j \in B)) \geq 0 \), while \( \{Y_n; n \geq 1\} \) is said to be \textit{negatively associated} (NA) if, for any disjoint finite subsets \( A, B \) of \( \{1, 2, \cdots\} \) and coordinatewise increasing functions \( f \) and \( g \), we have \( \text{Cov}(f(Y_i; i \in A), g(Y_j; j \in B)) \leq 0 \). The concept of PA was introduced in [5], while that of NA in [8].

Newman and Wright [12] and Su et al. [19] obtained the central limit theorem (CLT) for partial sums of PA or NA random variables as follows. Let \( \{Y_n; n \geq 1\} \) be a sequence of strictly stationary PA or NA random variables with \( E(Y_1) = 0 \), \( 0 < \text{Var}(Y_1) < \infty \) and \( S_n := \sum_{i=1}^{n} Y_i \). If

\[
\sigma^2 := \text{Var}(Y_1) + 2 \sum_{j=2}^{\infty} \text{Cov}(Y_1, Y_j) < \infty,
\]

then

\[
\frac{S_n}{\sqrt{n}} \xrightarrow{D} N(0, \sigma^2) \quad \text{as} \quad n \to \infty.
\]

This suggests that, for any \( 0 \leq \varepsilon < 1 \), there exist positive constants \( k_1 \) and \( k_2 \) such that

\[
k_1 e^{-\left(1+\varepsilon\right)x^2/2} \leq P\left\{ \frac{|S_n|}{\sqrt{n}\sigma} \geq x \right\} \leq k_2 e^{-\left(1-\varepsilon\right)x^2/2}, \quad x \geq 1,
\]

for sufficiently large \( n \). The inequality (1.4) represents upper and lower bounds of the tail probability (cf. Lemma 2 in page 175 of [6]).
Next, consider the case of mixing random variables. For any two $\sigma$-fields $\mathcal{A}$ and $\mathcal{B}$ in $(\Omega, \mathcal{F}, P)$, define the correlation

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \frac{|E(VW) - E(V)E(W)|}{(EV^2)^{1/2}(EW^2)^{1/2}},$$

where the sup is taken over all square-integrable random variables $V$ and $W$ which are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable, respectively. Let now $\{Y_n; n \geq 1\}$ be a sequence of strictly stationary random variables with $E(Y_1) = 0$ and $0 < \text{Var}(Y_1) < \infty$. For any nonempty disjoint sets $S$ and $D$ of $\{1, 2, \cdots\}$, denote

$$\rho(S, D) = \rho(\sigma[Y_i; i \in S], \sigma[Y_j; j \in D]),$$

where $\sigma[\cdot]$ is the $\sigma$-field generated by $Y_i$'s. The "distance" between any two disjoint nonempty subsets $S, D$ of $\{1, 2, \cdots\}$ will be denoted by $\text{dist}(S, D) := \min_{j \in S, k \in D} \|j - k\|$, where $\|\cdot\|$ is the usual Euclidean norm. For each $n \geq 1$, define $\rho_n^* = \sup \rho(S, D)$, where the sup is taken over all pairs of nonempty disjoint subsets $S, D$ of $\{1, 2, \cdots\}$ such that $\text{dist}(S, D) \geq n$. Let again $S_n = \sum_{i=1}^n Y_i$ and put $\sigma_n^2 = \text{Var}(S_n)$.

Peligrad [15] proved the following result: If $\rho_n^* \to 0$ (say, $\rho^*$-mixing) and $\sigma_n^2 \to \infty$ as $n \to \infty$, then we have (1.3) and (1.4) in this case as well under the condition (1.2).

In the next section, we study asymptotic properties for increments of partial sum processes of dependent random sequences under the assumption (2.2) in Section 2 which involves (1.1) for i.i.d. random variables and (1.4) for $\rho^*$-mixing, PA or NA dependent random variables.

### 2. Main Results

In this paper, we develop some limit results for increments of partial sum processes of iid random sequences given as in [3, 9, 10] to the case of dependent random sequences as follows. Let $\{\xi_j; j \geq 1\}$ be a centered strictly stationary random sequence with $E\xi_1^2 = 1$. Define

$$S_0 = 0, \quad S_n = \sum_{j=1}^n \xi_j \quad \text{and} \quad \sigma(n) = \sqrt{ES_n^2}. \quad (2.1)$$

Assume that $\sigma(n)$ can be extended to a continuous function $\sigma(t)$ of $t > 0$ which is nondecreasing and regularly varying with exponent $\alpha$ at $\infty$ for some $0 < \alpha < 1$.

A positive function $\sigma(t)$, $t > 0$, is said to be regularly varying with exponent $\alpha > 0$ at $b \geq 0$ if $\lim_{t \to b} \{\sigma(zt)/\sigma(t)\} = x^\alpha$ for all $x > 0$. 
On the basis of the result (1.4) obtained above for \( \rho^*\)-mixing, PA or NA random fields, in this paper, we suppose that there exists \( n_0 \geq 1 \) such that, for any \( n \geq n_0 \) and \( 0 \leq \varepsilon < 1 \), there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 e^{-(1+\varepsilon)x^2/2} \leq P \left\{ \frac{|S_n|}{\sigma(n)} \geq x \right\} \leq c_2 e^{-(1-\varepsilon)x^2/2}, \quad x \geq 1.
\]

It is well-known that, as \( n \to \infty \), \( V_n/\sigma(n) \xrightarrow{p} 1 \) in (1.1) and (2.2) for centered independent random variables under the Lindeberg condition (cf. [4]), and that \( \sigma(n)/\sqrt{n} \sigma \to 1 \) holds for standard deviations of \( S_n \) in (2.2) and \( S_n \) in (1.4) (cf. [14, 16, 20]).

Suppose that \( \{a_n, n \geq 1\} \) is a nondecreasing sequence of positive integers such that

(i) \( 1 \leq a_n \leq n \).

Denote

\[
\beta_n = \left\{ 2 \left( \log(n/a_n) + \log \log n \right) \right\}^{1/2}, \quad n > e.
\]

The main results are as follows.

**Theorem 2.1.** Let \( \{\xi_j; j \geq 1\} \) be a centered strictly stationary random sequence with \( E\xi_1^2 = 1 \) and condition (2.2), and let \( \{a_n, n \geq 1\} \) be a nondecreasing sequence of positive integers satisfying condition (i). Then we have

\[
\limsup_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} \leq 1 \quad \text{a.s.} \tag{2.3}
\]

In order to obtain the opposite inequality of (2.3), the conditions on \( a_n \) and \( \{\xi_j; j \geq 1\} \) are a little bit restricted as in Theorem 2.2 below.

A random sequence \( \{\xi_j; j \geq 1\} \) is said to be linearly negative quadrant dependent (LNQD) if, for any positive number \( \lambda_j \) and disjoint subsets \( A, B \) of \( \mathbb{Z}_+ \), the inequality

\[
P \left\{ \sum_{j \in A} \lambda_j \xi_j \geq x, \sum_{k \in B} \lambda_k \xi_k \geq y \right\} \leq P \left\{ \sum_{j \in A} \lambda_j \xi_j \geq x \right\} P \left\{ \sum_{k \in B} \lambda_k \xi_k \geq y \right\} \tag{2.4}
\]

holds for all real numbers \( x \) and \( y \). This definition of LNQD was introduced by Newman [11].

In general NA sequence is obviously LNQD, but LNQD sequence does not imply NA (cf. [13], [17]).
Theorem 2.2. Let \( \{\xi_j; j \geq 1\} \) and \( \{a_n, n \geq 1\} \) be as in Theorem 2.1. Further assume that

(ii) the random sequence \( \{\xi_j; j \geq 1\} \) is LNQD

and

(iii) \( \limsup_{n \to \infty} a_n/n =: \rho < 1 \).

Then we have

\[
\limsup_{n \to \infty} \frac{|S_{n+a_n} - S_n|}{\sigma(a_n)\beta_n} \geq 1 \quad \text{a.s.} \tag{2.5}
\]

Combining Theorems 2.1 and 2.2 yields the following lim sup result.

Corollary 2.1. Under the assumptions of Theorem 2.2, we have

\[
\limsup_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} = 1 \quad \text{a.s.}, \tag{2.6}
\]

\[
\limsup_{n \to \infty} \frac{|S_{n+a_n} - S_n|}{\sigma(a_n)\beta_n} = 1 \quad \text{a.s.} \tag{2.7}
\]

Next, we are to consider the liminf problem.

Theorem 2.3. Let \( \{\xi_j; j \geq 1\} \) and \( \{a_n, n \geq 1\} \) be as in Theorem 2.1. Further assume that

(iv) \( \lim_{n \to \infty} \log(n/a_n)/\log \log n = r, \quad 0 \leq r \leq \infty \).

Then we have

\[
\liminf_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} \leq \left( \frac{r}{1+r} \right)^{1/2} \quad \text{a.s.} \tag{2.7}
\]

In order to obtain the opposite inequality of (2.7), the condition on \( \{\xi_j; j \geq 1\} \) is restricted as in Theorem 2.4 below.

The random sequence \( \{\xi_j, j \geq 1\} \) is said to be \textit{negatively lower orthant dependent} (NLOD) if, for any disjoint finite subsets \( A_1, \cdots, A_n \) of \( \mathbb{Z}_+ \) and coordinatewise increasing functions \( f_1, \cdots, f_n \), we have

\[
P\{f_1(\xi_{i_1}; i_1 \in A_1) \leq x_1, \cdots, f_n(\xi_{i_n}; i_n \in A_n) \leq x_n\}
\leq \prod_{j=1}^{n} P\{f_j(\xi_{i_j}; i_j \in A_j) \leq x_j\}
\]

for all real numbers \( x_1, \cdots, x_n \). Note that NA sequence implies NLOD but the converse is not true in general (cf. [2], [7]).
Theorem 2.4. Let \( \{\xi_j; j \geq 1\} \) and \( \{a_n, n \geq 1\} \) be as in Theorem 2.3 with (iv). Further assume that either

(v) the random sequence \( \{\xi_j, j \geq 1\} \) is NLOD

or

(vi) \( P\left\{ \max_{1 \leq m \leq L} \frac{S_{[mr]+[r]} - S_{[mr]}}{\sigma([r])} \leq \sqrt{2 \log L} \right\} \leq c L^{-\delta_0}, \)

for some \( r \in \mathbb{R}_+ \) and \( \delta_0 > 0 \), where \( m, L \in \mathbb{Z}_+ \) and \( c \) is a positive constant. Then we have

\[
\liminf_{n \to \infty} \sup_{0 \leq i \leq n} \left| S_{i+a_n} - S_i \right| \sigma(a_n) \beta_n \geq \left( \frac{r}{1+r} \right)^{1/2} \quad \text{a.s.} \tag{2.8}
\]

For instance, a stationary condition for correlation functions of the Gaussian or Ornstein-Uhlenbeck random sequence \( \{\xi_j; j \in \mathbb{Z}_+\} \) yields the above condition (vi) (cf. Lemmas 2.3-2.4 in [1]). In this point of view, the latter implies a mild condition as contrasted with \( \rho^* \)-mixing, NA or PA random fields.

Combining Theorems 2.3 and 2.4 gives the following liminf result.

Corollary 2.2. Under the assumptions of Theorem 2.4, we have

\[
\liminf_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \left| S_{i+j} - S_i \right| \sigma(a_n) \beta_n = \left( \frac{r}{1+r} \right)^{1/2} \quad \text{a.s.} \tag{2.9}
\]

When \( 0 \leq r < \infty \) in condition (iv), the liminf result in (2.9) is different from the limsup result in (2.6). In order to obtain a limit result, we consider the following condition (iv)' of Theorem 2.5 when \( r = \infty \) in (iv). Noting that \( \lim_{n \to \infty} \frac{r}{1+r} = 1 \) at the right hand side of (2.8), it is immediate to obtain the following theorem.

Theorem 2.5. Let \( \{\xi_j; j \geq 1\} \) be a centered strictly stationary random sequence with \( E\xi_1^2 = 1 \) and condition (2.2), and let \( \{a_n, n \geq 1\} \) be a nondecreasing sequence of positive integers satisfying conditions (i) and

(iv)' \( \lim_{n \to \infty} \log(n/a_n)/\log \log n = \infty, \)

Further assume that either condition (v) or (vi) holds as in Theorem 2.4. Then we have

\[
\liminf_{n \to \infty} \sup_{0 \leq i \leq n} \left| S_{i+a_n} - S_i \right| \sigma(a_n) \beta_n \geq 1 \quad \text{a.s.} \tag{2.10}
\]

Combining Theorems 2.1 and 2.5, we arrive at the following limit result.
Corollary 2.3. Under the assumptions of Theorem 2.5, we have

\[
\lim_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} = 1 \quad \text{a.s.}
\]
\[
\lim_{n \to \infty} \sup_{0 \leq i \leq n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n)\beta_n} = 1 \quad \text{a.s.}
\]

(2.11)

The original versions of (2.11) are frequently called the Csörgő-Révész increments in literature (cf. Chapter 3 in [3]). We see that NA sequence implies LNQD as well as NLOD from previous statements. Thus, under NA condition, we have the following result.

Corollary 2.4. If the condition (ii) in Theorem 2.2 is replaced by

(vii) the random sequence \{ξ_j, j ≥ 1\} in Theorem 2.1 is NA,

then we have (2.6) as well. Also if the condition (v) in Theorem 2.4 is replaced by (vii), then we have (2.9) as well.

Example 2.1. Let \{ξ_j, j ≥ 1\} be an NA Gaussian random sequence in Corollary 2.3. Then the condition (2.2) is satisfied. Set \(a_n = [\log n]\). Then, the sequence \{a_n, n ≥ 1\} satisfies all the conditions of Theorems 2.2 and 2.5 with

\[
\beta_n = \left\{ 2 \left( \log(n/[\log n]) + \log \log n \right) \right\}^{1/2}.
\]

Thus we have, from (2.6) and (2.11),

\[
\lim_{n \to \infty} \sup_{i \leq n} \frac{|S_{n+[\log n]} - S_n|}{\sigma([\log n]) \sqrt{2 \log n}} = 1 \quad \text{a.s.,}
\]
\[
\lim_{n \to \infty} \sup_{0 \leq i \leq n} \frac{|S_{i+[\log n]} - S_i|}{\sigma([\log n]) \sqrt{2 \log n}} = 1 \quad \text{a.s.}
\]

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