\textbf{M/PH/1 QUEUE WITH DETERMINISTIC IMPATIENCE TIME}

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\textbf{ABSTRACT}

We consider an \textit{M/PH/1} queue with deterministic impatience time in which customers have phase-type service requirements. We find a related Markov process by using Markovian structure of the phase type distribution for services, and then obtain the stationary distribution of the Markov process. By using the results of the stationary distribution of the Markov process, we obtain performance measures such as the loss probability, the waiting time distribution and the queue size distribution.

1 INTRODUCTION

In many service systems, customers wait for service for a limited time only and leave the system if not served during that time. Such customers with limited waiting time are usually referred to as impatient customers. Impatience phenomena are often encountered in real-time communication systems, inventory systems with storage of perishable goods, telecommunication networks, call centers, etc. (See [6].)

Systems with limited waiting times can be classified as follows:
- The limitation acts only on waiting time or on sojourn time.
- The customer can calculate his prospective waiting time at the arrival epoch and balks if this exceeds his patience or he joins the queue regardless, leaving the system if and when his patience expires.

Combining these two distinctions, Baccelli et al. [2] describe the following four queueing systems with impatient customers:
(a) limitation on sojourn time (impatience until the end of service), aware customers: The entering customer leaves immediately if he knows that his total sojourn time is beyond his patience.
(b) limitation on sojourn time, unaware customers: This is the case if customers do not know anything about the system and are unaware of the beginning of the service.
(c) limitation on waiting time (impatience until the beginning of service), aware customers: The same as (a) above with the impatience acting only on waiting time.
(d) limitation on waiting time, unaware customers: The same as (b) above with the impatience acting only on waiting time.
In this paper we consider an $M/PH/1$ queue with deterministic impatience time in which customers have phase-type service requirements. For this queue, we deal with the case (d) mentioned above. We find a related Markov process by using Markovian structure of the phase type distribution for services, and then obtain the stationary distribution of the Markov process. By using the results of the stationary distribution of the Markov process, we obtain performance measures such as the loss probability, the waiting time distribution and the queue size distribution.

This paper is inspired by de Kok and Tijms [16] and Xiong et al. [20]. They also studied the $M/G/1 + D$ queue and obtained an integral equation for the waiting time distribution. However, analytical solution for this equation was given for only special cases of service time distributions such as exponential distributions and hyper-exponential distributions of order 2 [20]. For more general $M/G/1 + D$ queue, approximations for the loss probability and the mean waiting time were given [16,20]. Hence it turns out that obtaining an exact analytical expression for performance measures such as the loss probability, the waiting time distribution and the queue size distribution, is meaningful and interesting in our judgement in the case where service time distribution is much more general than exponential and hyper-exponential distributions. Hence we consider the case where service time has a phase-type distribution. Note that every distribution can be approximated arbitrarily closely by a phase-type distribution.

2 $M/PH/1$ QUEUE WITH DETERMINISTIC IMPATIENCE ON WAITING TIME

We consider an $M/PH/1$ queue with deterministic impatience time in which customers arrive according to a Poisson process with intensity $\lambda$. The service time has a phase-type distribution with representation $(\alpha, T)$ of order $m$ and mean $\mu^{-1}$.

In this section we deal with the case (d) mentioned in Section 1, where an arriving (unaware) customer enters the system and leaves if his attained waiting time exceeds the fixed time $\tau$.

2.1 Workload process

We denote by $V(t)$ the workload (unfinished work, or virtual waiting time) at time $t$. Clearly \(\{V(t) : t \geq 0\}\) is a Markov process. Further it is a regenerative process with returning points to 0 as regeneration epochs. The mean of a regeneration cycle is finite since customers who arrive when the unfinished work is larger than $\tau$ will be lost; and the mean service time is finite. In addition, the distribution of a regeneration cycle is nonlattice. Hence the workload process \(\{V(t) : t \geq 0\}\) has a limiting distribution, which is also a stationary distribution. Let $P(x)$, $x \in \mathbb{R}$, be the distribution function of the workload process in steady state.

**Lemma 1** There exists a density function $p(x)$, $x \geq 0$, such that

$$P(x) - P(0) = \int_0^x p(y) \, dy.$$

Let

$$p_0 \equiv P(0) = \lim_{t \to \infty} \mathbb{P}(V(t) = 0).$$

The stationary distribution of the workload process is given as follows.
Theorem 2 The $p_0$ and $p(x)$ are given by

$$
p_0 = \left(1 + \lambda \alpha \left\{ \int_0^\tau e^{(\lambda_1 \alpha + T)y} dy + e^{(\lambda_1 \alpha + T)\tau} (-T)^{-1} \right\} \right)^{-1},$$

$$
p(x) = \begin{cases} 
p_0 \lambda \alpha e^{(\lambda_1 \alpha + T)\tau} 1, & 0 < x \leq \tau, \\
p_0 \lambda \alpha e^{(\lambda_1 \alpha + T)\tau} e^{T(x-\tau)} 1, & x > \tau. \end{cases}
$$

2.2 Performance measures

We find the performance measures such as loss probability, waiting time distribution and queue size distribution, from the result of the stationary distribution of the workload process $\{V(t) : t \geq 0\}$.

Loss probability

The loss probability is the probability that the workload immediately before an arbitrary epoch is larger than $\tau$, which is $1 - P(\tau)$ by the `Poisson arrivals see time average' (PASTA) property. Therefore

$$
p_{\text{loss}} = 1 - P(\tau) = \int_\tau^\infty p(x) dx \\
= p_0 \lambda \alpha e^{(\lambda_1 \alpha + T)\tau} (-T)^{-1} 1.
$$

Waiting time distribution

Let $W$ denote the waiting time of an arbitrary customer who is served. Let

$$
w_0 \equiv \mathbb{P}(W = 0) \\
w(x) \equiv \frac{d}{dx} \mathbb{P}(W \leq x), \quad 0 < x < \tau.
$$

By PASTA, we have

$$
w_0 = \frac{p_0}{1 - p_{\text{loss}}} \\
w(x) = \frac{p(x)}{1 - p_{\text{loss}}}, \quad 0 < x < \tau.
$$

From this, moments of the waiting time $W$ of an arbitrary served customer can be calculated. Specifically,

$$
\mathbb{E}[W] = \frac{p_0 \lambda \alpha}{1 - p_{\text{loss}}} \int_0^\tau x e^{(\lambda_1 \alpha + T)\tau} dx 1, \\
\mathbb{E}[W^2] = \frac{p_0 \lambda \alpha}{1 - p_{\text{loss}}} \int_0^\tau x^2 e^{(\lambda_1 \alpha + T)\tau} dx 1.
$$

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