On Dispersion Managed Solitons

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ABSTRACT

We give a brief survey on dispersion managed non-linear Schrödinger equation which is used in fiber optics communications.

INTRODUCTION

The equation describing the amplitude \(u\) of a signal in a glass-fiber cable is given by a cubic non-linear Schrödinger equation (NLS)

\[
i \partial_t u = -d(t) \partial_x^2 u - c(t) |u|^2 u,
\]

where \(t\) stands for the distance along the fiber, \(x\) the time, \(d(t)\) the group velocity dispersion and \(c(t)\) the loss and amplification effects.

Its solution is related to a soliton which moves without change of its shape keeping a balance between dispersion and non-linearity. In 1980, Lin, Kogelnik and Cohen [4] proposed the idea of dispersion management technique for such a balance and it was very successful and has a record breaking transmission rates of more than 1 Tbits/s over an 18,000 kilometer optical fiber [6]. Furthermore, this technique is widely used commercially.

DISPERSION MANAGED SOLITONS

By rescaling of (1) we obtain

\[
i \partial_t u = -d_0(t) \partial_x^2 u - \epsilon(\alpha \partial_x^2 u + |u|^2 u),
\]

where \(d_0(t)\) is the mean zero component of group velocity dispersion, \(\alpha\) the average dispersion along one period and \(\epsilon\) a small parameter. Separating the free motion given by solution of \(i \partial_t u = -d_0(t) \partial_x^2 u\) and averaging over the period, Gabitov-Turytsin [1] obtained for the “averaged” solution \(v\)

\[
i \partial_t v = -\epsilon \alpha \partial_x^2 v - \epsilon Q(v, v, v),
\]

where \(Q(v_1, v_2, v_3) := \int_0^1 T_t^{-1} [T_t v_1 T_t v_2 T_t v_3] dt\) and \(T_t = e^{it \partial_x^2}\) is the solution operator of the free Schrödinger equation. Then by separation of variables, \(v(t, x) = e^{i \omega t} f(x)\), one can get the so-called dispersion managed solitons as solutions of

\[
-\omega f = -\alpha f'' - Q(f, f, f)
\]

(2)
which is the Euler-Lagrange equation for the averaged Hamiltonian

\[ H(f) = \frac{\alpha}{2} \int_{\mathbb{R}} |f'(x)|^2 \, dx - \frac{1}{4} Q(f, f, f, f), \]

where \( Q(f_1, f_2, f_3, f_4) := \int_0^1 \int_{\mathbb{R}} T_t f_1(x) T_t f_2(x) T_t f_3(x) T_t f_4(x) \, dx \, dt = \langle Q(f_1, f_2, f_3), f_4 \rangle. \)

**HISTORY**

Weak solutions of (2) depend on the sign of the average dispersion \( \alpha \).

When \( \alpha \) is positive, using \( H(f) < 0 \), one can show that \( \omega \) is also positive. Zharnitsky, Grenier, Jones and Turitsyn showed in [8] that there exists a weak solution of (2) in \( H^1 \) and then by bootstrapping it is in \( H^s \cap C^\infty \) for all \( s > 0 \). Indeed, they showed the existence of a minimizer of Hamiltonian \( H(f) \) with the constraint \( \|f\|_2^2 = \lambda \) for any function \( f \in H^1 \) and used Lions’ concentration compactness principle. For decay rate of solutions as \( x \to \infty \), there is a numerical conjecture [5] on exponential decay but no rigorous proof so far.

When \( \alpha = 0 \), it is easy to show \( \omega > 0 \). Due to lack of compactness, it is harder to show the existence of a solution. However, Kunze [3] showed the existence of a minimizer in \( L^2 \cap L^\infty \) and Stanislavova [7] showed Kunze’s minimizer is indeed in \( C^\infty \) using Bourgain spaces and Tao’s bilinear estimates. In this case also there is a numerical conjecture of exponential decay and the authors [2] showed rigorously that solutions decay super-algebraically, not exponentially. However this result is the first rigorous proof on this aspect.

For \( \alpha < 0 \), if there is a solution, \( \omega \) is also positive. But there is a conjecture that there is no solution for negative average dispersion which is not proved so far.

**REFERENCES**