A finite element method for the incompressible Navier-Stokes equations on non-convex polygons

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ABSTRACT

In this talk, we study a finite element method for the incompressible Navier-Stokes equations on non-convex polygonal domains. In a neighborhood of the non-convex corner the velocity and pressure functions have a decomposition into the singular part plus the regular part in the space $H^2 \times H^1$. We propose a numerical scheme for the regular part and the stress intensity factors, show the stability and derive the error estimates. Some numerical experiments are given.

INTRODUCTION

We consider the incompressible Navier-Stokes system

$$
\begin{align*}
-\mu \Delta u + (u \cdot \nabla)u + \nabla p &= f \quad \text{in } \Omega, \\
\text{div } u &= g \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma,
\end{align*}
$$

(1)

where $\Omega$ is a non-convex polygonal domain in $\mathbb{R}^2$ with the boundary $\Gamma := \partial \Omega$; $u$ is the velocity vector; $p$ is the pressure; $f, g$ are given functions.

We here describe the formulas of the singularity functions for the Stokes operator $L$ defined by $L[v, q] := [-\mu \Delta v + \nabla q, -\text{div } v]$. The singular exponents of the Stokes operator with no-slip condition are ordered as follows:

$$
1/2 < \lambda_1 < \pi/\omega < \Re \lambda_2 < \Re \lambda_3 < 2\pi/\omega < \cdots,
$$

where $\lambda_i$ are the roots of the algebraic equation: $\sin^2(\lambda_i \omega) - \lambda_i^2 \sin^2 \omega = 0$. Let $\chi = \chi(r) \in C^\infty(\mathbb{R}^2)$ be the cutoff function satisfying $\chi \equiv 1$ if $r \leq r_0$, and $\chi \equiv 0$ if $r \geq 3r_0$ for a number $r_0 > 0$. The corner singularity functions are defined by

$$
\Phi_i = \chi r^{-\lambda_i} T_i(\theta), \quad \phi_i = \chi r^{-\lambda_i-1} \xi_i(\theta),
$$

(2)

where $T_i(\theta)$ and $\xi_i(\theta)$ are the trigonometric functions corresponding the singular exponent $\lambda_i$, satisfying $T_i(\omega_1) = T_i(\omega_2) = 0$ with the rays $\theta = \omega_1, \omega_2$ away from the non-convex vertex. Using the eigenvalue $-\lambda_i$ instead of $\lambda_i$ in the singularity functions $\Phi_i$ and $\phi_i$, the dual singular functions are defined by

$$
\Phi_i^- = \chi r^{-\lambda_i} T_i^-(\theta), \quad \phi_i^- = \chi r^{-\lambda_i-1} \xi_i^-(\theta),
$$

(3)
where \( T_\theta^- (\theta) \) and \( \xi_\theta^- (\theta) \) are the trigonometric functions corresponding to the eigenvalue \(-\lambda_\theta\).

As above, the explicit form of the corner singularity function in a neighborhood of the corner is found. So, using the corner singularity function, the approximate solution pair \([u_h, p_h]\) is characterized by \( w_h, \sigma_h \) and \( C_{i,h} \), which has the following decomposition:

\[
\begin{align*}
\mathbf{u}_h &= \mathbf{w}_h + \sum_{i=1}^{2} C_{i,h} \Phi_i, \\
p_h &= \sigma_h + \sum_{i=1}^{2} C_{i,h} \phi_i.
\end{align*}
\]

Our aim is how to find the approximate coefficient \( C_{i,h} \) for \( i = 1, 2 \) and construct the approximate pair \([w_h, \sigma_h]\). We explain the method to find these functions and value as follows. Let \( V_h \) and \( M_h \) be finite dimensional subspaces of \( H^1_0(\Omega) \) and \( L^2(\Omega) \), which satisfy the inf-sup condition. Using the iterative scheme and the finite element method, we compute the approximate solution of the Navier-Stokes equations (1) with the corner singularity functions. Our algorithm of computing the finite element solution \([w_h^n, \sigma_h^n]\) and the approximate coefficients \( C_{i,h} \) for \( i = 1, 2 \) is as follows:

1. For \( i = 1, 2 \), find \([\Psi_{i,h}, \psi_{i,h}]\) \( \in V_h \times M_h \) such that

\[
\begin{align*}
a(\Psi_{i,h}, \mathbf{v}_h) - b(\psi_{i,h}, \mathbf{v}_h) &= -\langle \Gamma_{i,s}, \mathbf{v}_h \rangle \quad \forall \mathbf{v}_h \in V_h, \\
b(q_h, \Psi_{i,h}) &= \langle \gamma_{i,s}, q_h \rangle \quad \forall q_h \in M_h,
\end{align*}
\]

where \([\Gamma_{i,s}, \gamma_{i,s}] := L[\Phi_i^-, \phi_i^-] \).

2. Set \( n = 1 ; \ w_h^0 = 0 \) and \( C_{1,h}^0 = C_{2,h}^0 = 0 \).

3. Set \( u_h^{n-1} = w_h^{n-1} + C_{1,h}^{n-1} \Phi_1 + C_{2,h}^{n-1} \Phi_2 \).

4. Calculate

\[
\begin{align*}
C_{1,h}^n &= \gamma_1 \int_\Omega [- (u_h^{n-1} \cdot \nabla) u_h^{n-1} + f] \cdot (\Phi_1^- + \Psi_{1,h}) - g(\phi_1^- + \psi_{1,h}) \ dx, \\
C_{2,h}^n &= \gamma_2 \int_\Omega [- (u_h^{n-1} \cdot \nabla) u_h^{n-1} - C_{1,h}^n \Phi_1^+ + f] \cdot (\Phi_2^- + \Psi_{2,h}) \\
&\quad - (C_{1,h}^n g_{1,s} + g)(\phi_2^- + \psi_2^-) \ dx.
\end{align*}
\]

5. Find \([w_h^n, \sigma_h^n]\) \( \in V_h \times M_h \) such that

\[
\begin{align*}
a(w_h^n, \mathbf{v}_h) - b(\sigma_h^n, \mathbf{v}_h) &= \langle \Gamma_{2,h}^{n-1}, \mathbf{v}_h \rangle \quad \forall \mathbf{v}_h \in V_h, \\
b(q_h, w_h^n) &= \langle g_{2,h}^{n-1}, q_h \rangle \quad \forall q_h \in M_h,
\end{align*}
\]

where

\[
\begin{align*}
\Gamma_{2,h}^{n-1} &= - (u_h^{n-1} \cdot \nabla) u_h^{n-1} - C_{1,h}^n \Phi_1^s - C_{2,h}^n \Phi_2 + f, \\
g_{2,h}^{n-1} &= C_{1,h}^n g_{1,s} + C_{2,h}^n g_{2,s} + g.
\end{align*}
\]

6. Set \( n \leftarrow n + 1 \), and \( u_h^n = w_h^n + C_{1,h}^n \Phi_1 + C_{2,h}^n \Phi_2 \).

7. Repeat 3-6 until \( ||u_h^n - u_h^{n-1}||_0 < \text{TOL} \).

**NUMERICAL EXPERIMENTS**

Let \( \Omega \) be the L-shaped domain, which has the following form

\[
\Omega = ((-1, 1) \times (-1, 1)) \backslash ([1, 0] \times [1, 0]).
\]
The opening angle of the concave vertex is \( \omega = 3\pi/2 \). Let \((r, \theta)\) be the polar coordinate. The cutoff function \( \chi \in C_0^2(\mathbb{R}^2) \) is defined by \( \chi(r) = 1 \) for \( r \leq 1/4 \), \( \chi(r) = \zeta(r) \) for \( 1/4 \leq r \leq 3/4 \), and \( \chi(r) = 0 \) for \( 3/4 \leq r \), where \( \zeta(r) := -192r^5 + 480r^4 - 440r^3 + 180r^2 - 33.75r + 3.375 \). The singular exponents \( \lambda_i \) for \( i = 1, 2 \) are the roots of \( \sin^2(\lambda_i \omega) - \lambda_i^2 \sin^2 \omega = 0 \) such that \( 1/2 < \lambda_1 < \lambda_2 < 1 \). For the case \( \omega = 3\pi/2 \), these roots are approximated by \( \lambda_1 \approx 0.5445 \) and \( \lambda_2 \approx 0.9085 \). Using the cutoff function \( \chi \), the corner singularity functions \( \Phi_i \) and \( \phi_i \) for \( i = 1, 2 \) are defined by (2).

Set \( \mu = 1 \) in (1) for simplicity. We choose the exact solutions \( u, p \) of (1) of the following form:

\[
    u = w + C_1 \Phi_1 + C_2 \Phi_2, \quad p = \sigma + C_1 \phi_1 + C_2 \phi_2
\]

where \( w = (w_1, w_2) \), \( \sigma = xy(x - y) \) and \( C_1 = C_2 = 1 \) with \( w_1 = w_2 = (x - x^3)(y - y^3) \). The right hand side \( f = -\Delta u + (u \cdot \nabla)u + \nabla p \) and \( g = \text{div } u \) directly calculated by (4) are given. Using the given functions \( f \) and \( g \), the approximate coefficient \( C_{i,h}^N \) and the discrete solution \( [w_h^N, \sigma_h^N] \) are computed, where \( N \) is the number of iterations with \( \text{TOL} = 10^{-7} \). Let \( u_h^N \) and \( p_h^N \) be the approximate functions defined by \( u_h^N := w_h^N + \sum_{i=1}^2 C_{i,h}^N \Phi_i \) and \( p_h^N := \sigma_h^N + \sum_{i=1}^2 C_{i,h}^N \phi_i \).

To confirm the efficiency of our algorithm, we also find the finite element approximate pair \([\tilde{u}_h^N, \tilde{p}_h^N]\) obtained by the usual finite element method in the same finite dimensional subspace. In Figure 1-3, we compare the approximate pairs computed by two different methods.

![Figure 1](image1.png)

(a) \(|u_1^N - u_{1,h}^N|\)

(b) \(|u_1^N - \tilde{u}_{1,h}^N|\)

Figure 1. Graphs of the errors for \( u_{1,h}^N \) and \( \tilde{u}_{1,h}^N \).

![Figure 2](image2.png)

(a) \(|u_2^N - u_{2,h}^N|\)

(b) \(|u_2^N - \tilde{u}_{2,h}^N|\)

Figure 2. Graphs of the errors for \( u_{2,h}^N \) and \( \tilde{u}_{2,h}^N \).
Figure 3. Graphs of the errors for $p_N^h$ and $\tilde{p}_N^h$.

Figure 4. Comparison of stream lines.

REFERENCES


