Fractional dynamical systems using the multistage generalized differential transform method

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ABSTRACT
In recent work, author in [1] employed the modified differential transform method(MDTM) for solving fractional Chen system. He derived numerical solutions by the MDTM with fractional order $\alpha = 0.99$ and compared the results with ones obtained by the generalized Adams-Bashforth-Moulton method(GABMM), which is well know method for solving fractional problems. In this work, we show that MDTM does not give a reliable approximate solutions with a low order of fractional derivative. We analyse why the MDTM does not work well and propose the new multistage generalized differential transform method(MsGDTM) to overcome the difficulties arising in the MDTM. Illustrative examples show the performance of MsGDTM.

DESCRIPTION OF THE MULTISTAGE GENERALIZED DTM

Let us consider the following fractional initial value problem:

$$D^\alpha y(t) = f(t, y(t)), \quad t > 0, \quad y(t_0) = y_0.$$  (1)

The basic idea of the MsGDTM is to apply the standard DTM to the problem in each sub-domain. To describe the MsGDTM we consider the equally spaced partition $P: 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ where the nodes $t_i = i \cdot h, \; h = T/N$. On the $i-$th sub-domain $\Omega_i \equiv (t_i, t_{i+1}), \; i = 0, 1, \cdots, N - 1$, we define $y(t)|_{\Omega_i} \equiv y_i(t)$. The generalized differential transform $Y_i(k)$ of $y_i(t)$ at $t = t_i$ is defined by

$$Y_i(k) = \frac{1}{\Gamma(\alpha k + 1)}[(D^\alpha)^k y_i(t)]_{t=t_i}.$$

The generalized differential inverse transform of $Y_i(k)$ is defined by

$$y_i(t) = \sum_{k=0}^{\infty} Y_i(k)(t - t_i)^{\alpha k}.$$
Suppose that $y_i(t)$ can be approximated by the $n$-partial sum

$$y_i(t) \approx \sum_{k=0}^{n} Y_i(k)(t - t_i)^{\alpha k} \equiv s_{i,n}(t).$$

Some fundamental properties of GDTM give the following recursive relation for the differential transform

$$\frac{\Gamma(\alpha(k+1) + 1)}{\Gamma(\alpha k + 1)} Y_i(k+1) = F_i(k), \quad k = 0, 1, 2, \ldots,$$

where $F_i(k)$ is the generalized differential transform of $f(t, y(t))$ at $t = t_i$.

The updating initial condition in the standard MsGDTM gives the value of $y(t_p)$, $p > 1$ as follows

$$y(t_p) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} (t_{j+1} - \tau)^{\alpha-1} f(s_{j,n}(\tau)) d\tau.$$

However, the proposed method gives the value of $y(t_p)$, $p > 1$ as follows

$$y(t_p) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} (t_p - \tau)^{\alpha-1} f(s_{j,n}(\tau)) d\tau,$$

which is close to the exact integral form. On both methods there are the same values of $y(t_p)$ for $p = 1$. However, it is easy to see that the standard MsGDTM gives to get more inaccurate approximate solution for large $p$.

**NUMERICAL ILLUSTRATION**

In this section, we present the numerical results of the following fractional Chen system

$$D^\alpha x = a(y - x),$$

$$D^\alpha y = (c - a)x - xz + cy,$$

$$D^\alpha z = xy - bz,$$

$$x(0) = c_1, y(0) = c_2, z(0) = c_3,$$

where $D^\alpha$ is the Caputo fractional derivative, $a$, $b$, $c$ are constants and $0 < \alpha \leq 1$.

**Theorem 0.1** ([3], [4]) Assume that $x^*$ be the equilibrium point and $M$ is the lowest common multiple of the $u_i$. Then, the equilibrium point $x^*$ is asymptotically stable if $|arg(\lambda)| > \pi/(2M)$ for all $\lambda$ which are the solution of the equation

$$\det(diag([\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \ldots, \lambda^{M\alpha_n}]) - J(x^*) = 0$$

where the Jacobian matrix $J(x^*) = \partial f / \partial x |_{x^*}$

$(\pi/2M) - \min_i |arg(\lambda_i)|$ is called the Instability measure for equilibrium points (IMFOS) in fractional order systems. Then, IMFOS < 0 implies that the system becomes asymptotically stable and attracts the nearby trajectories.

In this part, we set $a = 35$, $b = 3$, $c = 28$ and we take the initial conditions $x(0) = -10$, $y(0) = 0$ and $z(0) = 37$. This system has three equilibria and positive equilibrium pt $x^*$ is
\[ x^* = (\sqrt{(2c - a)b}, \sqrt{(2c - a)b}, 2c - a) = (7.9393, 7.9393, 21) \]  

Figure 1 shows IMFOS for the fractional order \(0 < \alpha < 1\).

![Figure 1. The IMFOS of the fractional Chen system](image)

It is easy to see that the fractional Chen system is asymptotically stable for \(\alpha < 0.824\). For \(\alpha = 0.8\), Figure 2 shows the phase diagrams obtained by MDTM, MsGDTM and GABMM, respectively.

![Figure 2. The phase plot](image)

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**REFERENCES**
