MAPPED WENO SCHEMES BASED ON A NEW SMOOTHNESS INDICATOR FOR HAMILTON-JACOBI EQUATIONS

Youngsoo Ha 1, Chang Ho Kim 2, Yeon Ju Lee 3 and Jungho Yoon 3

1) National Institute for Mathematical Sciences, Daejeon, 305-811, KOREA
2) Department of Computer Engineering, Glocal Campus, Konkuk University, 380-701, Chungju, KOREA
3) Department of Mathematics, Ewha W. University, Seoul, 120-750, KOREA

ABSTRACT

In this paper, we introduce an improved version of mapped weighted essentially nonoscillatory (WENO) schemes for solving Hamilton-Jacobi equations. To this end, we first discuss new smoothness indicators for WENO construction. Then the new smoothness indicators are combined with the mapping function developed by Henrick et. al. [24]. The proposed scheme yields fifth-order accuracy in smooth regions and sharply resolve discontinuities in the derivatives. Numerical experiments are provided to demonstrate the performance of the proposed schemes on a variety of one-dimensional and two-dimensional problems.

INTRODUCTION

In this paper, we consider numerical solutions for the Hamilton-Jacobi (H-J) partial differential equations of the form

\[ \phi_t + H(\nabla \phi) = 0, \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d, \]

\[ \phi(x, 0) = \phi_0(x) \]

where \( H \) is the Hamiltonian. In recent years, such equations have attracted from such areas as the optimal control, image processing, seismic waves, crystal growth, robotic navigation, geometric optics, differential games and calculus of variations. The equation(1) is a global nonlinear first-order problem, and it is well known that it does not have classical solutions even though the Hamiltonian and boundary conditions are smooth. Solutions of H-J equations with smooth initial condition are continuous but may develop discontinuities in a finite time.

To ensure the existence of the solution as well as to single out the physically relevant solution, the introduction of the notions of entropy conditions and viscosity solutions was needed for H-J equations. Such development is due to Crandall, Evans and Lions among many others. To find the (correct) solution of (1), commonly used entropy condition is the vanishing viscosity technique. Souganidis [22] established the convergence of general approximation schemes to this equation. Osher and Sethian [15] used the connection between conservation laws and H-J equations to construct higher order accurate numerical methods. Most of the numerical ideas
are similar to hyperbolic conservation laws and the H-J equations, theoretical and numerical concepts used for conservation laws can be extended to H-J equations.

In [16], Osher and Shu constructed high order essentially nonoscillatory (ENO) schemes for solving the H-J equations. ENO schemes for solving the H-J equations on unstructured meshes were constructed by Lafon and Osher [11]. Solving the H-J equations based on weighted ENO schemes developed by Liu et al. [14] and Jiang and Shu [8] presented by Jiang and Peng in [7]. After that, WENO schemes have been used successfully in designing high-order schemes for H-J equations such as weighted central ENO schemes in [2], Hermite WENO schemes in [17], weighted power ENO schemes in [18], mapped WENO schemes in [3], high-order WENO schemes on unstructured meshes [23]. The central high resolution schemes proposed by Kurganov and Tadmor [10] and Lin and Tadmor [13]. There are other approaches for approximating solutions of H-J equations such as discontinuous Galerkin methods [6,12] and relaxation schemes [9].

High-order methods based on essentially nonoscillatory (ENO) schemes [4,5,19,20] were introduced by Osher and Shu [16], which is evolved in time with a monotone flux. The main concept of ENO scheme is to use an adaptive stencil based on the local smoothness such that it avoids interpolation across discontinuities. To this end, a smoothness indicator of a solution is first determined over each stencil and then, by using this, the smoothest one is chosen from a set of candidate stencils. As a result, it obtains information from smooth regions and avoids spurious oscillations near discontinuities. The WENO scheme, an improved version of the ENO technique using a cell-averaged approach, was introduced by Liu, Osher, and Chan [14] keeping the robustness and high-order accuracy of ENO schemes. WENO uses a nonlinear convex combination of all the candidate stencils to improve the accuracy of numerical fluxes while maintaining the non-oscillatory behavior of ENO schemes near discontinuities. This process is performed by weighting the contribution of the local flux according to its smoothness on each stencil such that the weight of the solution on a stencil containing a discontinuity is essentially zero. In doing this, the corresponding WENO scheme can achieve a high order accuracy without oscillations near discontinuities or sharp gradient regions. In [7], Jiang and Shu introduced a smoothness indicator, which is the sum of the normalized squares of the scaled $L^2$ norms of the all derivatives of the lower order polynomials, and then extended a finite difference (flux) version of WENO schemes [14] to third- and fifth-order accurate methods. However, it has been noted by Henrick, Aslam, and Powers [24] that the fifth-order WENO-JS scheme provides only third-order accuracy near critical points where the first derivative of the solution vanishes. To fix this deficiency, they presented the Mapped WENO scheme by using a simple nonlinear mapping to the WENO-JS weights such that the resulting scheme can achieve the optimal order of accuracy near critical points.

In this work, we propose a new smoothness indicator and combine them with the mapping function developed by Henrick et. al. [24]. The fifth convergence order of the resulting Mapped WENO scheme (hereafter, denoted by WENO-NSM) near critical points is immediately obtained due to the convergence of the weights to the ideal weights (which generate the central upstream fifth-order scheme for the 5-points stencil) at critical points. In this study, the WENO-NSM is especially applied for the solution of Hamilton-Jacobi equation. In fact, the Mapped WENO method in [24] was used for the solution of Hamilton-Jacobi equation for the first time by Bryson and Levy [1]. However, numerical results confirm that the WENO-NSM scheme provides improved behavior over the Mapped WENO method [1].
REFERENCES


