ASYMPTOTIC SYNCHRONOUS BEHAVIOR OF THE KURAMOTO TYPE MODELS WITH FRUSTRATIONS

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ABSTRACT

We present a quantitative asymptotic behavior of coupled Kuramoto oscillators with frustrations and give some sufficient conditions for the parameters and initial condition leading to phase or frequency synchronization. We consider three ensembles of Kuramoto oscillators with frustration. First, we study a general case with nonidentical oscillators; i.e., the natural frequencies are distributed. Second, as a special case, we study an ensemble of two groups of identical oscillators. When two identical Kuramoto oscillator groups are mixed, we focus on the synchronous stage, which describes the relaxation of phase synchronization in each group from a segregation state. Finally, we consider a Kuramoto-type model that was recently derived from the Van der Pol equations for two coupled oscillator systems in the work of Lèck and Pikovsky [1]. We provide a framework in which the phase synchronization of each group is attained. Moreover, the constant frustration causes the two groups to segregate from each other, although they have the same natural frequency.

THE KURAMOTO MODEL WITH FRUSTRATIONS

The purpose of this paper is to study the dynamic interplay between distinct natural frequencies \((\text{intrinsic frustration})\) and phase shift in interactions \((\text{interaction frustration})\) among Kuramoto oscillators. More precisely, we present several sufficient conditions for the complete (frequency) synchronization in terms of the initial phase diameter, the (interaction) frustration, and the coupling strength. Synchronization is ubiquitous in various disciplines such as physics, biology, chemistry, and the social sciences [2]. However, rigorous mathematical treatments of synchronization were initiated only a few decades ago by two pioneers; namely, Winfree [3] and Kuramoto [4,5], who introduced simple ODE models for limit-cycle oscillators. Kuramoto and Sakaguchi [6] proposed a variant of the Kuramoto model in which the coupling function incorporated frustration (phase shift) so that richer dynamical phenomena would be observed than with no frustration. Let \(\theta_i = \theta_i(t)\) be the phase of the \(i\)th Kuramoto oscillator. Further, let \(\alpha_{ji}\) be the frustration between the \(j\)th and \(i\)th oscillators, which is assumed to be symmetric in \(i\) and \(j\). In this situation, the dynamics of Kuramoto oscillators is governed by the following ODE system:

\[
\dot{\theta}_i = \Omega_i + \frac{K}{N} \sum_{i=1}^{N} \sin(\theta_j - \theta_i + \alpha_{ji}), \quad -\frac{\pi}{2} < \alpha_{ji} < \frac{\pi}{2},
\]  \(^{(1)}\)
where $\Omega_i$ is a natural frequency of the $i$th oscillator, $K$ denotes the positive coupling strength, and $N$ denotes the number of oscillators. Each natural frequency is a random variable extracted from some given density function $g = g(\Omega)$. Note that the R.H.S. of (1) is Lipschitz continuous, so the well-posedness of the system (1) is well known from the Cauchy-Lipschitz theory. Thus, what matters about the solutions is the dynamic behavior such as the relaxation process, the shape of phase-locked states, and the existence of global attractors. In general, the frustration hinders synchronization, so coupling strength greater than that of the original Kuramoto model without frustration is needed to guarantee global synchronization. The use of frustration is needed for modeling real physical and biological systems, but it causes numerous mathematical difficulties in analyzing the synchronization. By varying the value of $\alpha$ in a series of numerical simulations, Zheng [7] investigated how frustration can induce a desynchronization of oscillators. Daido [8] observed that frustration is common in disordered interactions. Recently, the effect of frustration has also been intensively studied in relation to networks of oscillators [9–11]. Levnajić [12] introduced the notion of link frustration to characterize and quantify the dynamical states of networks. In particular, Montbrió et al. [13] analyzed synchronization between two interacting populations of different phase oscillators. Even though some studies have been conducted for systems with frustration, these were mostly based on the numerical approach. Furthermore, one of the most important reasons for the interest in frustration is that the Kuramoto model with frustration has no conservation law, even for identical oscillators. Thus, the standard energy method [14–19] based on a conservation law cannot be applied in our setting. Hence, analyzing the large-time behavior of physical systems without conservation laws is challenging, and to the best knowledge of the authors there are so far no general tools for dealing with such systems.

In this paper, we consider three Kuramoto-type models with frustration. First, we study the general case of nonidentical Kuramoto oscillators with natural frequencies that are distributed. Second, as a special case, we deal with an ensemble consisting of two groups of identical oscillators. When two identical Kuramoto oscillator groups are mixed, the whole configuration evolves into the segregated state and then asymptotically toward the phase-locked state. Finally, we consider a Kuramoto-type model that was recently derived from the Van der Pol equations for two coupled oscillator systems in the work of Lück and Pikovsky [1].

REFERENCES


